Factorization and Division in the Realm of Linear Ordinary BVPs

$$\boxed{\begin{array}{l} u'' = f \\ u(0) = u(1) = 0 \end{array}} = \boxed{\begin{array}{l} ? \\ ? \end{array}} \cdot \boxed{\begin{array}{l} ? \\ ? \end{array}}$$

$$(-AX - XB + XAX + XBX)(A - FA)^{-1} = A$$
$$(A - FA)^{-1} = ?$$

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RICAM

Part I:
Factorization

$$\boxed{\begin{array}{l} u'' = f \\ u(0) = u(1) = 0 \end{array}} = \boxed{\begin{array}{l} ? \\ ? \end{array}} \cdot \boxed{\begin{array}{l} ? \\ ? \end{array}}$$

How Do We Factor BVPs?

$$\boxed{\begin{array}{l} u'' = f \\ u(0) = u(1) = 0 \end{array}} = \boxed{\begin{array}{l} u' = f \\ \int_0^1 u(\xi)\, d\xi = 0 \end{array}} \cdot \boxed{\begin{array}{l} u' = f \\ u(0) = 0 \end{array}}$$

$$[D^2, L \oplus R] = [D, F] \cdot [D, L]$$

Relation to Green's Operators?

What Is a BVP?

$$\boxed{\begin{array}{l} Tu = f \\ B_1 u = 0, \ldots B_n u = 0 \end{array}}$$

$T \in \mathbb{C}[\partial] \quad \mathrm{ord}(T) = n$

$B_i : \mathfrak{F} \to \mathbb{C} \quad \mathfrak{F} = C^\infty[0,1]$

$$B = B_1 \oplus \cdots \oplus B_n$$

$B : \mathfrak{F} \to \mathbb{C}^n \quad \tilde{B} : \mathfrak{F} \to \mathbb{C}^m$

$$B \oplus \tilde{B} : \mathfrak{F} \to \mathbb{C}^n \oplus \mathbb{C}^m$$
$$x \mapsto Bx + \tilde{B}x$$

$$\boxed{\begin{array}{l} Tu = f \\ Bu = 0 \end{array}}$$

A BVP is (T, B)

Abstract Setting: $T \in L(V) \quad \dim \mathrm{Ker}(T) = n < \infty$

$B : V \to K^n$ linear

When Are Two BVPs Equal?

$(T, B) \sim (\tilde{T}, \tilde{B})$ iff

$T = \tilde{T}$ and $\mathsf{Ker}(B) = \mathsf{Ker}(\tilde{B})$

$[T, B]$ denotes the equivalence class of (T, B)

Example: $Du \equiv u' \quad Lu \equiv u(0) \quad Ru \equiv u(1)$

$$[D^2, L \oplus R] = [D^2, (R - L) \oplus L]$$

$$\boxed{\begin{array}{l} u'' = f \\ u(0) = 0, u(1) = 0 \end{array}} \sim \boxed{\begin{array}{l} u'' = f \\ u(0) = u(1), u(0) = 0 \end{array}}$$

What Is a Regular BVP?

$$\begin{aligned} Tu &= f \\ Bu &= 0 \end{aligned}$$

$[T, B]$ is *regular* iff

$$\underset{f \in \mathfrak{F}}{\forall} \; \underset{u \in \mathfrak{F}}{\exists!} \; (Tu = f, Bu = 0) \Leftrightarrow$$

$\mathrm{Ker}(B) \cap \mathrm{Ker}(T) = 0$ and $\mathrm{Ker}(B) + \mathrm{Ker}(T) = \mathfrak{F}$

Uniqueness Existence

$$\Leftrightarrow \qquad \mathrm{Ker}(B) \dotplus \mathrm{Ker}(T) = \mathfrak{F} \qquad \Leftrightarrow$$

$\varphi_1, \ldots \varphi_n$ basis of $\mathrm{Ker}(T)$ (fundamental system)

$$\begin{pmatrix} B_1\varphi_1 & \cdots & B_1\varphi_n \\ \vdots & \ddots & \vdots \\ B_n\varphi_1 & \cdots & B_n\varphi_n \end{pmatrix}$$ is regular

The Green's Operator

BVP

$[T, B]$

$$\boxed{\begin{aligned} Tu &= f \\ Bu &= 0 \end{aligned}}$$

Green's Operator

$$G : \mathfrak{F} \to \mathfrak{F}$$
$$f \mapsto u$$

G solves $[T, B]$ iff $TG = 1$ and $BG = 0$

Regularity $\Rightarrow G$ is well-defined

Notation $G = [T, B]^{-1}$

General Formula:

$$[T, B]^{-1} = (1 - P)T^{\blacklozenge}$$

$P^2 = P$ projection

$\mathrm{Im}(P) = \mathrm{Ker}(T)$

$\mathrm{Ker}(P) = \mathrm{Ker}(B)$

$TT^{\blacklozenge} = 1$ a right inverse

$T^{\blacklozenge} = [T, L \oplus \cdots \oplus LD^{n-1}]^{-1}$

How Do We Multiply BVPs?

$$[T, B] \cdot [\tilde{T}, \tilde{B}] \equiv [T\tilde{T}, B\tilde{T} \oplus \tilde{B}]$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

$$G \qquad\qquad \tilde{G} \qquad \Rightarrow \qquad \tilde{G} \circ G$$

Proof: $TG = 1 \quad \tilde{T}\tilde{G} = 1$

$BG = 0 \quad \tilde{B}\tilde{G} = 0$

$$T\tilde{T}\tilde{G}G = T1G = TG = 1$$

$$B\tilde{T}\tilde{G}G = BG = 0 \qquad \tilde{B}\tilde{G}G = 0$$

"Notation" $([T, B] \cdot [\tilde{T}, \tilde{B}])^{-1} = [\tilde{T}, \tilde{B}]^{-1} \cdot [T, B]^{-1}$

$$[T, B], [\tilde{T}, \tilde{B}] \quad \Rightarrow \quad [T, B] \cdot [\tilde{T}, \tilde{B}]$$

regular $\qquad\qquad$ regular

The Problem Monoid

$$[T, B] \cdot [\tilde{T}, \tilde{B}] = [T\tilde{T}, B\tilde{T} \oplus \tilde{B}]$$ Multiplication

$$[1, 0]$$ $$\boxed{\begin{array}{l} 1u = f \\ 0u = 0 \end{array}}$$ Neutral Element

$$[T, B] \cdot [\tilde{T}, \tilde{B}] \neq [\tilde{T}, \tilde{B}] \cdot [T, B]$$ noncommutative

$$[1, 0]^{-1} = 1$$ Identity operator

Examples: $$\begin{array}{c} [D, L] \cdot [D, R] = [D^2, LD \oplus R] \\ \neq \\ [D, R] \cdot [D, L] = [D^2, RD \oplus L] \end{array}$$

IVP are commutative

$$[D, L] \cdot [D, L] = [D^2, L \oplus LD]$$

Elementary Green's Operators I

$$Af(x) \equiv \int_0^x f(\xi)\,d\xi \quad [D, L]^{-1} = A \qquad \begin{aligned} DA &= 1 \\ LA &= 0 \end{aligned}$$

$$Bf(x) \equiv \int_x^1 f(\xi)\,d\xi \quad [D, R]^{-1} = -B \qquad \begin{aligned} D(-B) &= 1 \\ R(-B) &= 0 \end{aligned}$$

$$\lceil f \rceil u \equiv fu \qquad [D - \lambda, L]^{-1} = \lceil e^{\lambda x} \rceil A \lceil e^{-\lambda x} \rceil$$

$$[D - \lambda, R]^{-1} = -\lceil e^{\lambda x} \rceil B \lceil e^{-\lambda x} \rceil$$

Elementary Green's Operators II

$$F \equiv A + B$$

$$Fu = \int_0^1 u(\xi)\, d\xi$$

$$[D, F]^{-1} = A - FA \equiv C$$

$\beta : \mathfrak{F} \longrightarrow \mathbb{C}$ linear $\quad [D - \lambda, \beta]$ regular, $\quad \beta(e^{\lambda x}) \neq 0$

$$[D - \lambda, \beta]^{-1} = (1 - P_{\lambda,\beta}) \lceil e^{\lambda x} \rceil A \lceil e^{-\lambda x} \rceil$$

$$P_{\lambda,\beta} \equiv \frac{\beta(u)}{\beta(e^{\lambda x})} e^{\lambda x}$$

Stieltjes Boundary Conditions

$$\beta(u) = \sum_{i=0}^{n-1} \Big(a_i u^{(i)}(0) + b_i u^{(i)}(1)\Big) + \int_0^1 \varphi(\xi) u(\xi)\, d\xi$$

$$\sum_{i=0}^{n-1} (a_i L D^i + b_i R D^i) + F\lceil \varphi \rceil$$

In $L, R, D, A, B, \lceil \varphi \rceil$ language:

$\beta \in$ right ideal generated by $\{L, R\}$

Multiplying BVPs and Identities for Green's Operators

$$
\begin{aligned}
[D,F]\cdot[D,L] &= [D^2, FD \oplus L] \qquad\qquad FDu = \int_0^1 u'(\xi)\,d\xi \\
&= [D^2, (R-L) \oplus L] \\
&= [D^2, L \oplus R] \qquad = [D,F]\cdot[D,R]
\end{aligned}
$$

$$
A\cdot(A - AF) = \begin{matrix} -A\lceil x\rceil - \lceil x\rceil B \\ +\lceil x\rceil A\lceil x\rceil + \lceil x\rceil B\lceil x\rceil \end{matrix} = (-B)\cdot(A - AF)
$$

How Do We Get Factorizations?

We want to factor	$[D^2, L \oplus R]$	
Choose a (regular) factor	$[D, L]$	$[D, R]$
Remaining factor	$[D, R]$	$[D, L]$
Compute	$[D, L]^{-1} = A$	$[D, R]^{-1} = -B$
Factorization	$[D, RA] \cdot [D, L]$	$[D, -LB] \cdot [D, R]$
	$= [D, F] \cdot [D, L]$	$= [D, F] \cdot [D, R]$

The Factorization Lemma

$[T, B]$ regular $\quad T = T_1 T_2$

Then there exist B_1, B_2 with

$[T_1, B_1], [T_2, B_2]$ regular and $\mathsf{Ker}(B) \leq \mathsf{Ker}(B_2)$

such that

$$[T, B] = [T_1, B_1] \cdot [T_2, B_2]$$

"Every factorization of the differential operator can be lifted to the problem level"

Splitting into Regular First-order Factors

$[T, B]$ regular $T = (D - \lambda_1) \cdots (D - \lambda_n)$

Then we can compute Stieltjes boundary conditions $\beta_1, \ldots, \beta_n$ such that

$$[T, B] = [D - \lambda_1, \beta_1] \cdots [D - \lambda_n, \beta_n]$$

"Every regular BVP can be factored into regular first-order BVPs"

Green's operators:

$$[T, B]^{-1} = [D - \lambda_n, \beta_n]^{-1} \cdots [D - \lambda_1, \beta_1]^{-1}$$

Part II:
Division in the Realm of Linear Ordinary BVPs

$$(-AX - XB + XAX + XBX)(A - FA)^{-1} = A$$
$$(A - FA)^{-1} = ?$$

From Green's Operators to Green's Functions

Problem $[T, B_1 \oplus \cdots \oplus B_n] \rightarrow$ Green's Operator G

$$\boxed{\begin{array}{l} TG = 1 \\ B_1 G = \ldots = B_n G = 0 \end{array}} \qquad \begin{array}{rl} G: & \mathfrak{F} \rightarrow \mathfrak{F} \\ & f \mapsto u \end{array}$$

Representation via Green's function:

$$u(x) = Gf\,(x) = \int_0^1 g(x,\xi) f(\xi)\, d\xi$$

$$\rightarrow \quad \mathfrak{G} \equiv \{g \mid g \text{ Green's Function}\}$$

Example:

$$\boxed{\begin{array}{l} u'' = f \\ u(0) = u(1) = 0 \end{array}} \quad g(x,\xi) = \begin{cases} (x-1)\xi & \text{if} \quad 0 \leq \xi \leq x \leq 1 \\ x(\xi - 1) & \text{if} \quad 0 \leq x \leq \xi \leq 1 \end{cases}$$

$$G = -AX - XB + XAX + XBX$$

Volterra's Kernel Composition

Volterra (1913): For $g, \tilde{g} \in \mathfrak{K}$ put:

$$g * \tilde{g}\,(x,y) = \int_0^1 g(x,t)\tilde{g}(t,y)\,dt$$

Noncommutative Ring $\mathfrak{K} \equiv \mathsf{L}^2(I \times I) \supseteq \mathfrak{G}$

Intention: $G \triangleq g, \tilde{G} \triangleq \tilde{g} \;\rightarrow\; G \circ \tilde{G} \triangleq g * \tilde{g}$

<u>Notation:</u> Often we identify $G \triangleq g$ and drop $*$.

Noncommutativity crucial: $AB \neq BA$

Factorization on Three Levels – An Example

Problem Level:

$$[D^2, L \oplus R] = [D, F] \cdot [D, L]$$

Operator Level:

$$\underbrace{-AX - XB + XAX + XBX}_{G_2} = A \cdot \underbrace{(A - FA)}_{C}$$

Functional Level:

$$\begin{aligned} &-h(\xi - x)\, x - h(x - \xi)\, x + h(\xi - x)\, x\xi + h(x - \xi)\, x\xi \\ &\quad = h(\xi - x) * \Big(-h(x - \xi) + h(\xi - x)\xi + h(x - \xi)\xi\Big) \end{aligned}$$

Factorization versus Division

Factorizations yield Divisions:

$$(-AX - XB + XAX + XBX)(A - FA)^{-1} = A$$

But what if Divisibility Fails?

$$(A - FA)^{-1} = ?$$

Recall the Integers:

$$6 \cdot 2^{-1} = 3 \in \mathbb{Z}$$
$$2^{-1} = 0.5 \in \mathbb{Q}$$

Mikusiński's Convolution Field

Mikusiński (1959): For $u, \tilde{u} \in \mathfrak{L}$ put:

$$u \circledast \tilde{u}\,(x) = \int_0^x u(x-\xi)\tilde{u}(\xi)\,d\xi$$

Commutative Ring $\mathfrak{L} \equiv C(0, \infty)$

Integral Operator $l \equiv 1$, so: $l \circledast u\,(x) = \int_0^x u(\xi)\,d\xi$

We have A but not B: We cannot solve BVPs!

Construct $\mathfrak{M}$ as the field of fractions of $\mathfrak{L}$.

Introduce «Differential Operator»: $s \equiv l^{-1}$

Solving Inhomogeneous IVPs à la Mikusiński

Fundamental Formula of Mikusiński Calculus:

$$s \circledast u = u' + u(0)\,\delta_0$$

$$s \circledast s \circledast u = u'' + u'(0)\,\delta_0 + u(0)\,\delta_0'$$

Dirac «Distribution»:

$$\delta_0 \equiv s \circledast 1 = f f^{-1} \quad \longrightarrow \quad \delta_0 \circledast u = u, \quad l \circledast \delta_0 = 1$$

Example:

$$\boxed{\begin{array}{l} u'' = f \\ u(0) = a, u'(0) = b \end{array}}$$

$$\begin{aligned} & s \circledast s \circledast u = f + a\,\delta_0 + b\,\delta_0' \\ \rightarrow\ & u = (l \circledast l) \circledast f + a\,(l \circledast 1) + b \\ & = x \circledast f + ax + b \end{aligned}$$

Localization in Noncommutative Rings

For localizing R at $S \subseteq R$ into RS^{-1}, we require:

Multiplicativity: $(\forall s, \tilde{s} \in S)\ s\tilde{s} \in S$

Ore Condition: $(\forall r \in R)(\forall s \in S)(\exists \tilde{r} \in R)(\exists \tilde{s} \in S)\, r\tilde{s} = s\tilde{r}$

Reversibility: $(\forall r \in R)\big((\exists s \in S)\, sr = 0 \Rightarrow (\exists \tilde{s} \in S)\, r\tilde{s} = 0\big)$

> *Necessary and sufficient for representing all elements of RS^{-1} as rs^{-1}: ring of fractions.*
>
> *Even if R has no zero divisors, it may fail to have a field of fractions (quotient field)!*

Motiviation for Ore Condition:

$$s^{-1}r = \tilde{r}\tilde{s}^{-1} \quad \rightarrow \quad r\tilde{s} = s\tilde{r}$$

The Ore Ring of Green's Functions

Applying the Construction:

First Attempt: $R = \mathfrak{K}$, $S =$ nonzerodiv of $\mathfrak{K}$

Second Attempt: $R = \mathfrak{K}$, $S = \langle A, B \rangle$

Third Attempt: $R = \mathfrak{K}$, $S = \mathfrak{G}$

Ore Condition tough!

Winning Idea:

Let R be any ring and S and multiplicative subset fulfilling the Ore condition. Then the ring S^+ generated by S in R fulfills the Ore condition when localized at S.

Final Choice: $R = \mathfrak{G}^+$, $S = \mathfrak{G}$

The Problem Monoid

Problem Monoid:

$$\mathfrak{B} = \{[T, B] \mid [T, B] \text{ Regular BVP}\}$$

Crucial Observation: $(\mathfrak{G}, \circ) \cong (\mathfrak{B}^{\mathrm{op}}, \cdot)$

Regularization Lemma:

For every T with $\mathrm{ord}(T) = m$ and every $B = B_1 \oplus \cdots \oplus B_m$ there is a $[\tilde{T}, \tilde{B}] \in \mathfrak{B}$ with $T|\tilde{T}$ and $\mathrm{Ker}(B) \geq \mathrm{Ker}(\tilde{B})$.

Division Lemma:

For every $[T, B], [T_1, B_1] \in \mathfrak{B}$ with $T_1|T$ and $\mathrm{Ker}(B) \leq \mathrm{Ker}(B_1)$ there is a unique $[T_2, B_2] \in \mathfrak{B}$ with $[T_1, B_1] \cdot [T_2, B_2] = [T, B]$.

The Ore Condition on Problems

Ore Condition in $\mathfrak{B}$:

Given $[T_1, B_1], [T_2, B_2] \in \mathfrak{B}$

Find $[\tilde{T}_1, \tilde{B}_1], [\tilde{T}_2, \tilde{B}_2] \in \mathfrak{B}$
such that $[T_1, B_1]\cdot[\tilde{T}_1, \tilde{B}_1] = [T_2, B_2]\cdot[\tilde{T}_2, \tilde{B}_2]$

Proof:

Regularization Lemma $\rightarrow [T, B] \in \mathfrak{B}$
with $T_1 T_2 | T$ and $\mathrm{Ker}(B) \leq \mathrm{Ker}(B_1 \oplus B_2)$

Division Lemma $\rightarrow [\tilde{T}_1, \tilde{B}_1], [\tilde{T}_2, \tilde{B}_2] \in \mathfrak{B}$

Lots of Fundamental Formulae

The Fundamental Formulae à la Mikusiński:

$$AD = 1 - L \qquad -BD = 1 - R$$

$$Au' = u - u(0) \qquad -Bu' = u - u(1)$$

$$u' = A^{-1}u - u(0)\,A^{-1}1 \qquad u' = -B^{-1}u + u(1)\,B^{-1}1$$

$$\boxed{A^{-1}u = u' + u(0)\,\delta_0} \qquad \boxed{B^{-1}u = -u' + u(1)\,\delta_1}$$

$$\delta_0 \equiv A^{-1}1 \nearrow \qquad \delta_1 \equiv B^{-1}1 \nearrow$$

Example of a Different Fundamental Formula:

$$CD = 1 - F$$

$$Cu' = u - \textstyle\int_0^1 u(\xi)\,d\xi$$

$$u' = C^{-1}u - \left(\textstyle\int_0^1 u(\xi)\,d\xi\right) C^{-1}1$$

$$\boxed{C^{-1}u = u' + \left(\textstyle\int_0^1 u(\xi)\,d\xi\right)\varepsilon}$$

$$\varepsilon \equiv C^{-1}1 \nearrow$$

Solving Inhomogeneous BVPs à la Mikusiński

Recall:

$$[D^2, L \oplus R] = [D, F] \cdot [D, L]$$
$$G_2 = A \cdot C$$

A Custom-tailored Fundamental Formula:

$$G_2^{-1} u = u'' + u(0)\, \delta_0' + u(1)\, \varepsilon$$
$$\delta_0' \equiv A^{-2} 1 \nearrow$$

Example:

$u'' = f$ $u(0) = a, u(1) = b$

$$G_2^{-1} u = f + a\, \delta_0' + b\, \varepsilon$$
$$\to u \overset{*}{=} G_2 f + a\, A(\delta_0 - 1) + b\, A1$$
$$= G_2 f + a\,(1 - x) + b\, x$$

* Uses $C\delta_0' = \delta_0 - 1$.

Conclusion

- Factorization of any regular BVP into irreducible factors
- Mikusiński calculus extended to cover boundary conditions

- Consider generalizations: variable coefficients, systems, PDEs
- Algorithmic tools from F1301 for noncommutative polynomial computation?
- Possible hybrid approach, e.g. fundamental system numerically