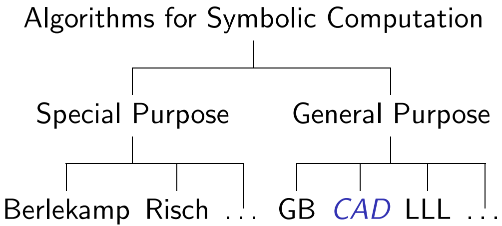
Symbolic Computation for Inequalities

Manuel Kauers

Orientation



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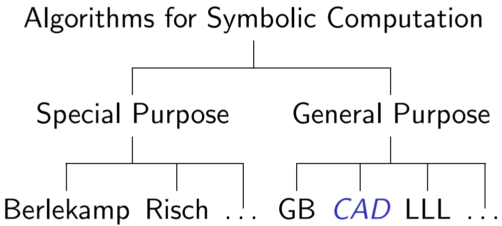

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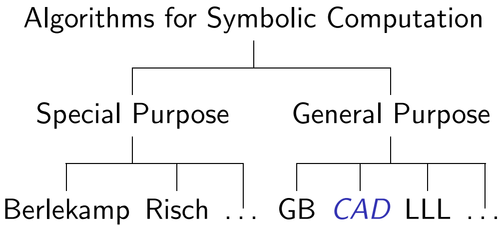

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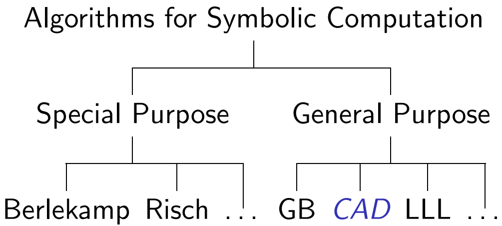

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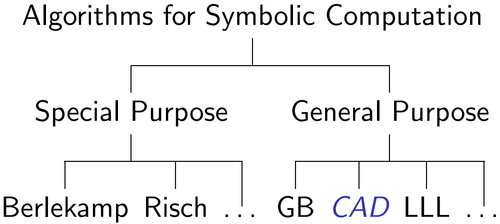

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- Many seemingly unrelated questions concerning inequalities can be answered by CAD.
- Main questions in this tutorial:
  When to apply CAD? and *How to apply CAD?*
- Not: How does CAD work.

I Typical Questions involving Inequalities

Question 1: Is this true?

Example: Let $a, b, c \in \mathbb{R}$ be such that

1. $a > 0, b > 0, c > 0$ and
2. $a + b + c = abc$.

Show that

$$\frac{1}{\sqrt{a^2+1}} + \frac{1}{\sqrt{b^2+1}} + \frac{1}{\sqrt{c^2+1}} \leq \frac{3}{2}.$$

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(Gröbner basis analog: Ideal membership)

Question 2: What are the solutions?

Example: Find all $x, y, z \in \mathbb{R}$ such that

$$\begin{aligned}
0 &= -10x^4 + 24yx^3 + 33x^3 - 16y^2x^2 + 5yx^2 - x^2 - 37y^2x \\
&\quad - 50yx - 22x + 2y^4 + 15y^3 - 61y^2 - 46y + 60 \\
0 &= -3x^4 + 7yx^3 + 10x^3 - 5y^2x^2 + 3yx^2 - x^2 + y^3x \\
&\quad - 12y^2x - 16yx - 6x + 7y^3 - 19y^2 - 18y + 20 \\
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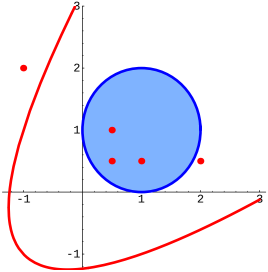

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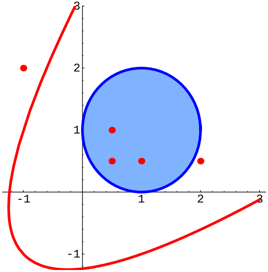

There are finitely many solutions: $(\frac{1}{2}, 1)$, $(1, \frac{1}{2})$, and $(\frac{1}{2}, \frac{1}{2})$.

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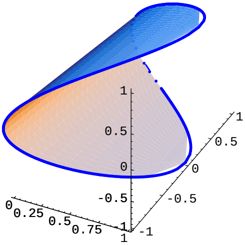
Special case: *real* solutions of algebraic equation systems
(Gröbner bases give complex solutions by default)

Question 3: What is the dimension?

Example 1: The real solution set $S \subseteq \mathbb{R}^3$ of the system

$$0 \leq x \leq 1,\ y^2 \leq 1 - x,\ z^2 = x$$

has dimension 2.



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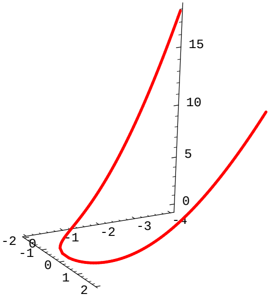

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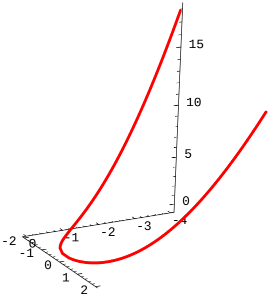

Problem Pattern: Determine the (real) dimension of the solution set of a system of polynomial inequalities

Question 4: When is this true?

Example: For which $a, b \in \mathbb{R}$ does the formula

$$\forall x, y \in \mathbb{R} : a^2 - 2b^2a - 2ya + (1-2a)x^2 + 4y^2 + x(2y - 4ba - 2a) \geq 0$$

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(Gröbner basis analog: Elimination)

II The Machine:

C*ylindrical* **A***lgebraic* **D***ecomposition*

(George E. Collins, 1975)

Algebraic Decomposition

A finite set of polynomials $\{p_1, \ldots, p_m\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$ induces a *decomposition* ("partition") of $\mathbb{R}^n$ into maximal sign-invariant *cells* ("regions").

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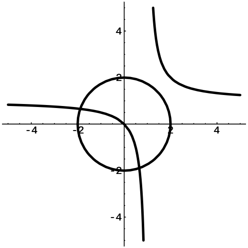

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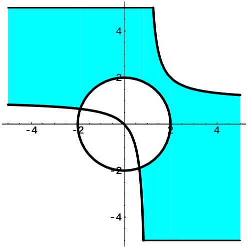

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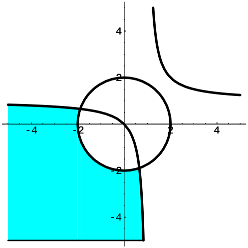

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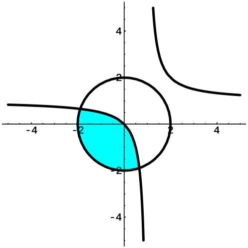

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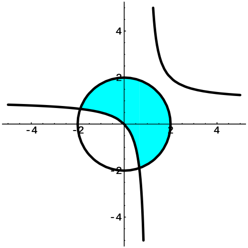

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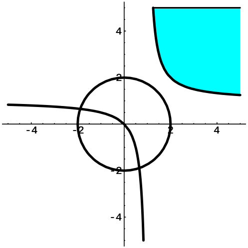

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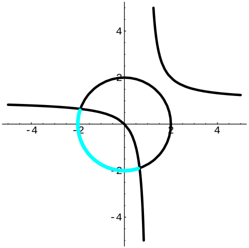

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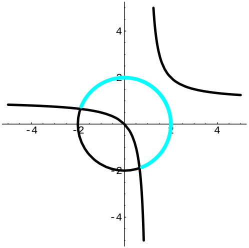

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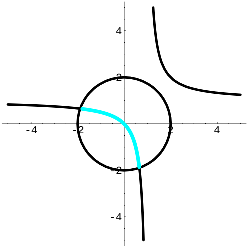

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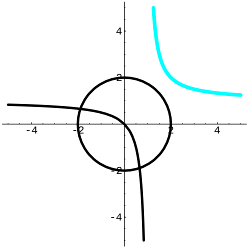

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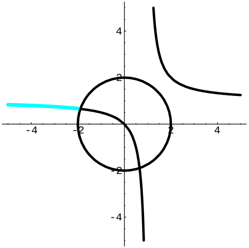

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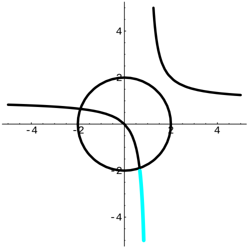

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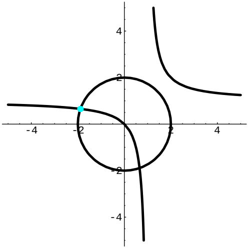

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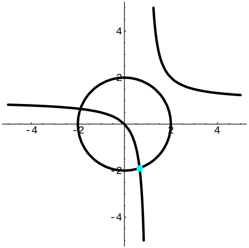

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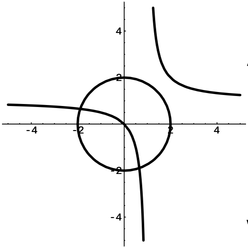
Example: The polynomials $p_1 = x^2 + y^2 - 4$ and $p_2 = (x-1)(y-1) - 1$ induce a decomposition of $\mathbb{R}^2$ into 13 cells:



Precise Definition:
A *cell* in the algebraic decomposition of

$$\{p_1, \ldots, p_m\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$$

is a *maximal connected* subset of $\mathbb{R}^n$ on which all the p_i are *sign invariant.*

Tarski Formulas

A Tarski Formula is a formula in *first order predicate logic* whose atomic formulas are of the form

$$\text{poly}(x_1, \ldots, x_n) \diamondsuit 0$$

where

- $\text{poly}(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$, and
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Examples

- $x^2 + y^2 \leq 1 \wedge (x-1)(y-1) > 1$
- $\forall\, x\ \exists\, y : x^2 + y^2 > z^2 \Rightarrow z^2 < 1$
- $\exists\, y : y^2 - x^5 < 0$

Tarski Formulas vs. Algebraic Decomposition

Truth of a Tarski Formula can be determined *by inspection* from the algebraic decomposition of the involved polynomials.

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Consider the cell(s) for which the quantifier free part

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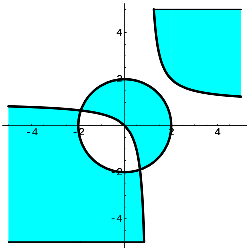
is true.



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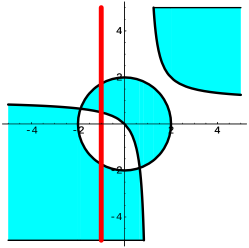
Obviously, each vertical line $x = \alpha$ intersects one of those cells nontrivially. The $\forall x \exists y$ claim follows.



Cylindrical Algebraic Decomposition: Motivation

Observation: It does not hurt if we change from a decomposition for $\{p_1, \ldots, p_m\}$ to a decomposition for $\{p_1, \ldots, p_m, q_1, \ldots, q_k\}$ for some polynomials $q_1, \ldots, q_k \in \mathbb{Q}[x_1, \ldots, x_n]$.

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This motivates the definition of a *Cylindrical Algebraic Decomposition.*

Cylindrical Algebraic Decomposition: Definition

For $n \in \mathbb{N}$, let

$$\pi_n \colon \mathbb{R}^n \to \mathbb{R}^{n-1}, \qquad (x_1, \dots, x_{n-1}, x_n) \mapsto (x_1, \dots, x_{n-1})$$

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Base case: Any algebraic decomposition of $\mathbb{R}^1$ is cylindrical.

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- A data structure for representing a cylindrical algebraic decomposition by a symbolic description and a sample point for each cell.
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These three notions are used in parallel, but this does usually not cause much confusion.

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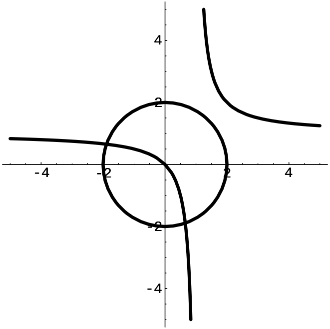

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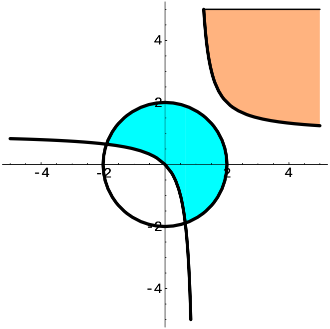

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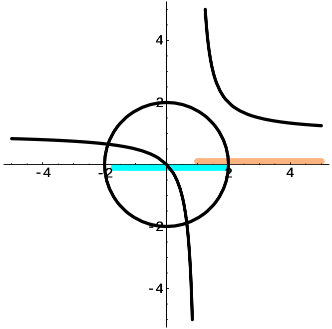

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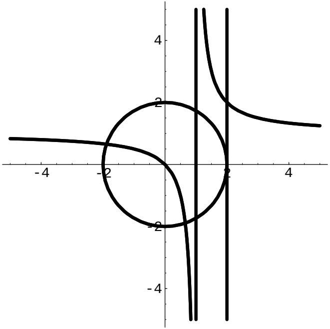

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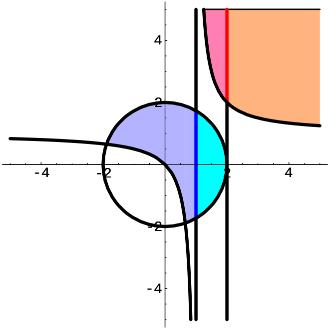

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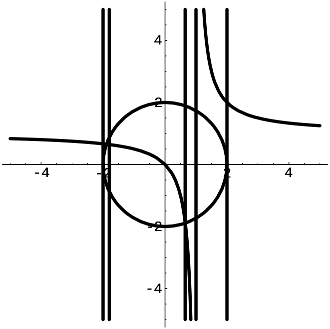

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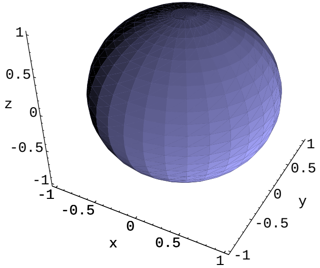
Proceed analogously for all other cell pairs. The result is a CAD.

Cylindrical Algebraic Decomposition: 3D-Example

Consider $\{x^2 + y^2 + z^2 - 1\} \subseteq \mathbb{Q}[x, y, z]$



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Consider $\{x^2 + y^2 + z^2 - 1\} \subseteq \mathbb{Q}[x, y, z]$

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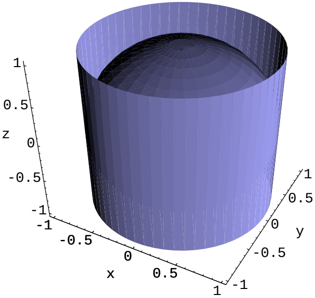

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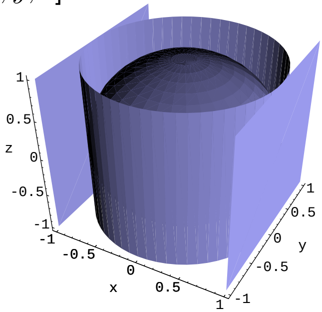

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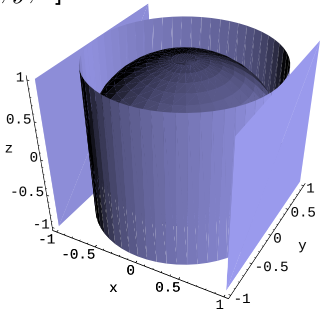

Result: $\{x^2 + y^2 + z^2 - 1, x^2 + y^2 - 1, x^2 - 1\}$ is a CAD for $\{x^2 + y^2 + z^2 - 1\}$.

Representation of a CAD in the Computer

The CAD-algorithm delivers a CAD in form of a *tree.*

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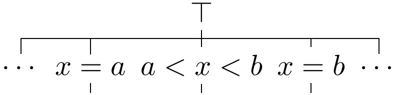

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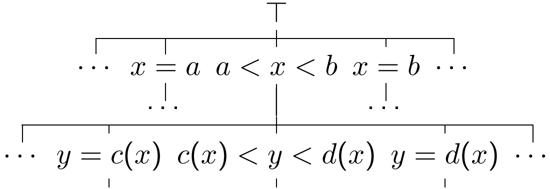

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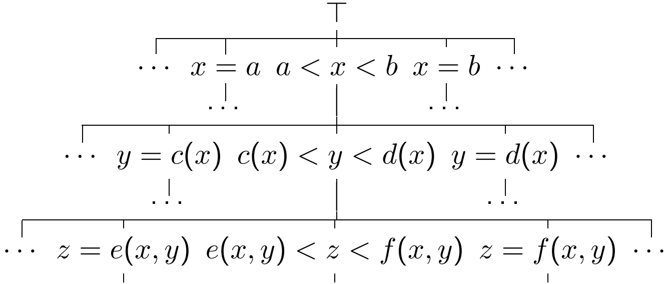

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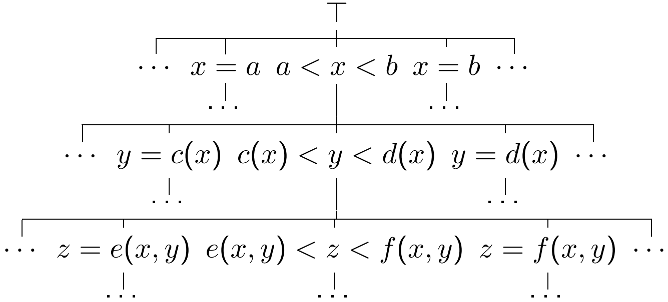

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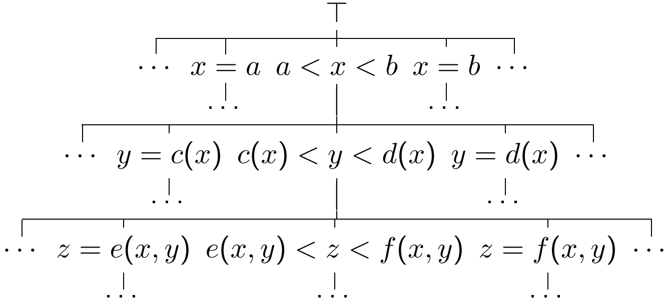

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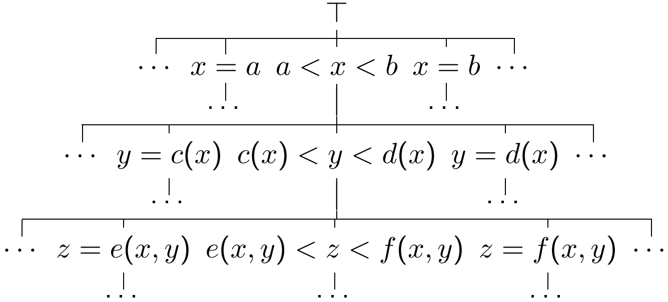

Each path in this tree describes an individual cell of the CAD. Sample points for each cell are easily obtained from this representation.

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Δ Δ Δ Δ Δ Δ Δ Δ Δ Δ Δ

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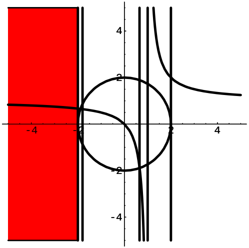

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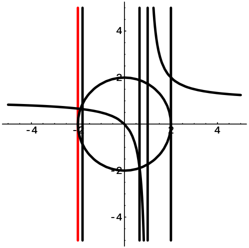

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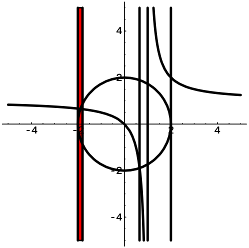

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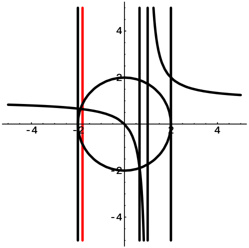

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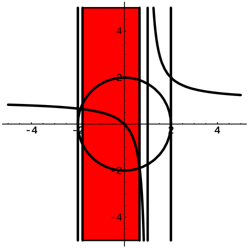

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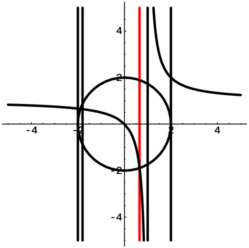

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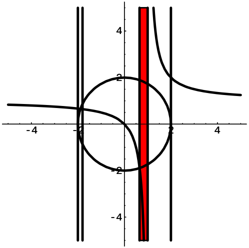

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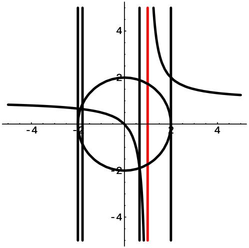

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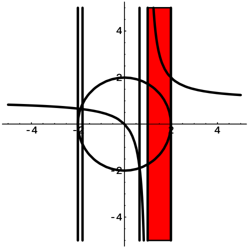

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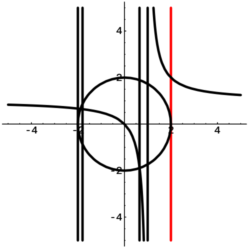

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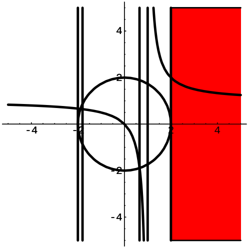

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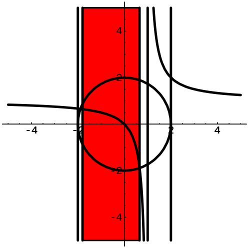

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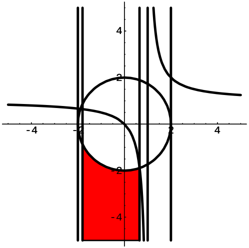

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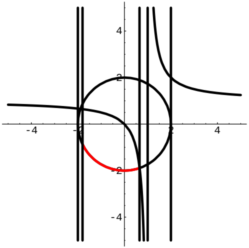

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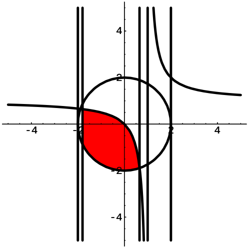

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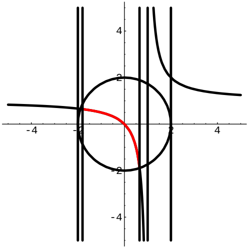

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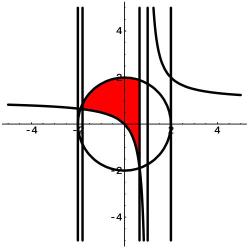

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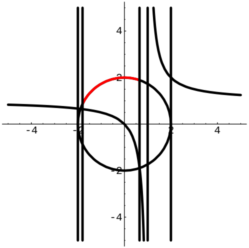

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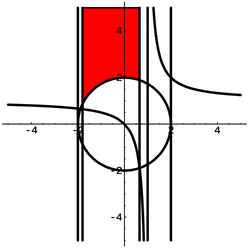

III Using CAD for Answering Questions involving Inequalities

Question 4: When is this true?

Problem Pattern: Given a Tarski formula

$$\Phi \equiv \forall\exists\, x_1, x_2, \ldots, x_n \in \mathbb{R} : A(x_1, \ldots, x_n, y_1, \ldots, y_m),$$

determine a *quantifier free* formula $B(y_1, \ldots, y_m)$ which is equivalent to Φ

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- Determine the subtrees where the quantifiers for $x_1, \ldots, x_n$ are respected. Replace them by 'true' and the others by 'false'.
- Return the disjunction of all path-conjunctions as $B(y_1, \ldots, y_m)$

The other Questions

Problem Pattern: Decide whether a given inequality is a consequence of some given constraints

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$\longrightarrow$ Homework

IV Example Applications of CAD

Proving Non-Polynomial Things

- Consider the Tarski fromula

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  - Setting $x = F_n$, $y = F_{n+1}$, $z = F_{n+2}$, we obtain

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  - This gives the *induction step* of an induction proof.
  - To complete the proof, just verify $F_1 > 0$, $F_2 > 0$.
- This simple application of CAD is strong enough to prove a lot of inequalities about quantities that satisfy *recurrence equations.*

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$$1 - \sum_{k=1}^{n} a_k < \prod_{k=1}^{n} (1 - a_k) < \frac{1}{1 + \sum_{k=1}^n a_k}$$

$$1 + \sum_{k=1}^{n} a_k < \prod_{k=1}^{n} (a_k + 1) < \frac{1}{1 - \sum_{k=1}^n a_k}$$

for all $n \geq 1$.

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> *Lesson:* Problems concerning nonpolynomial inequalities may be reduced to questions about polynomial inequalities that can be answered with CAD.

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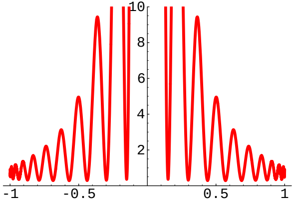
- The sum is heavily oscillating. The plot shows the case $n = 20$.



Proving Non-Polynomial Things – Warning

- S. Gerhold was able to derive *asymptotic envelopes* for $f_n(x)$:

$$f_n(x) = A(x) + 2|B(x)| \sin\big(2n\pi\theta(x) + \varphi(x)\big) + \mathrm{O}(\frac{\log n}{n})$$

where $\theta(x), \varphi(x)$ are irrelevant and $A(x)$ and $B(x)$ are complicated.

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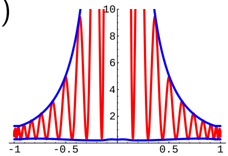
$$f_n(x) = A(x) + 2|B(x)| \sin\bigl(2n\pi\theta(x) + \varphi(x)\bigr) + \mathrm{O}(\frac{\log n}{n})$$

where $\theta(x), \varphi(x)$ are irrelevant and $A(x)$ and $B(x)$ are complicated.

The plot shows $A(x) \pm 2|B(x)|$.



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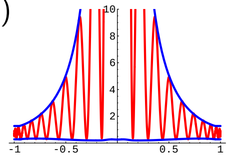

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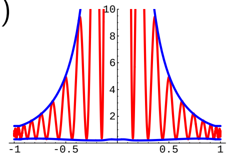

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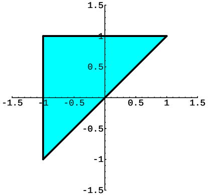
Lesson: Special function inequalities can be *very difficult.* (As opposed to identities. . .)

A Question asked by an Analysis Student

Question: What is the image of the triangle $(-1,-1), (-1,1), (1,1)$ under the map

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto (x^2+y^2, xy-1)\,?$$



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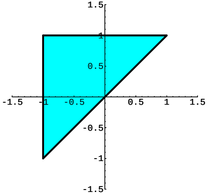

$$\overset{f}{\longmapsto} \qquad ?$$

Answer: Eliminate x, y from the formula

$$\exists x, y : (-1 \le x \le 1 \land -1 \le y \le 1 \land x \le y \land$$
$$X = x^2 + y^2 \land Y = xy - 1)$$

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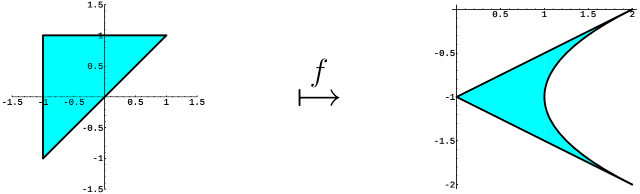


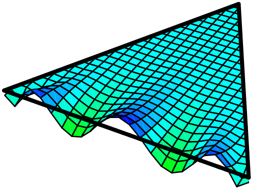
Result:

$$f(\Delta) = \{(x,y) \in \mathbb{R}^2 : \big(0 \le x \le 1 \wedge |y+1| \le \tfrac{1}{2}x\big) \\ \vee \big(1 < x \le 2 \wedge \sqrt{x-1} \le |y+1| \le \tfrac{1}{2}x\big)\}\}$$

Another Problem of Schöberl's

Find a polynomial $v \in \mathbb{R}[x, y]$ of total degree n with

- $v(x, 0) = \int_{-1}^{x} P_{n-1}(t)\, dt$
- $v(x, 1 - x) = v(x, 1 + x) = 0$



such that

$$\int_0^1 \int_{y-1}^{1-y} y\Big(\big(\frac{\partial}{\partial x} v(x, y)\big)^2 + \big(\frac{\partial}{\partial y} v(x, y)\big)^2\Big)\, dx\, dy$$

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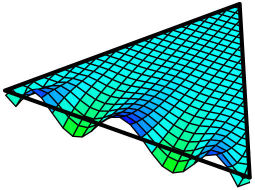

such that

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This problem is *open* for general n, but *easy* for specific n.

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Solution: Because of the constraints, the polynomials $v(x,y)$ are of the form

$$v(x,y) = (x-y+1)(x+y-1)\big(\int_{-1}^{x} P_{n-1}(t)\,dt/(x^2-1) + y\cdot\tilde{v}(x,y)\big).$$

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Applying CAD to this equation gives a formula

$$I = \min \land \left(a_{0,0} = u, a_{1,0} = v, \dots\right) \lor I > \min \land (\dots)$$

from which the coefficients can be extracted.

Further Applications of CAD

There are further applications of CAD in the SFB...

- ...in control theory (S. Ratschan, phase 1),
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→ Ask the colleagues for details if you are interested.

V What You Also Need to Know

Complexity

Warning! Computing a CAD for a system of m polynomials in n variables with total degree d may cost up to

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arithmetic operations runtime.

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deg	#vars	CAD	Lag. + GB
5	2	0.03s	0.02s
6	6	298.7s	0.05s
7	6	419.7s	0.07s
26	156	–	293.7s
27	156	–	331.2s

Unlike for Gröbner bases, this worst case bound is *often* experienced in practice.

Example: Runtime for computing $v(x, y)$ in Schöberl's problem.

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- Ask a specialist for help

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- *Mathematica:* part of the standard distribution from Version 5 on. Command names:
  - `CylindricalDecomposition` and
  - `Reduce`

The End