An Algebraic Foundation for Factoring Linear Boundary Problems *

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Abstract

Motivated by boundary problems for linear ordinary and partial differential equations, we define an abstract boundary problem as a pair consisting of a surjective linear map (representing the differential operator) and a subspace of the dual space (specifying the boundary conditions). This subspace is finite dimensional in the ordinary case, but infinite dimensional for partial differential equations. For so-called regular boundary problems, the given operator has a unique right inverse (called the Green's operator) satisfying the boundary conditions.

The main idea of our approach consists in the passage from a single problem to a compositional structure on boundary problems. We define the composition of boundary problems such that it corresponds to the composition of their Green's operators in reverse order. If the defining operators are endomorphisms, we can interpret the composition as the multiplication in a semidirect product of certain monoids. Given a factorization of the linear operator defining the problem, we characterize and construct all factorizations of a boundary problem into two factors.

In the setting of differential equations, the factor problems have lower order and are often easier to solve. For the case of ordinary differential equations, all the main results can be made algorithmic (in particular the determination of the factor problems). As a first example for partial differential equations, we conclude with a factorization of a boundary problem for the wave equation.

1 Introduction

To motivate our algebraic setting and the terminology, let us first recall the classical setting of a *two-point boundary problem* on a finite interval, see for

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example Stakgold [19]. Such a problem is given by a linear ordinary differential equation and boundary conditions describing linear combinations of some values of the solution and its derivatives at both endpoints of the interval. For example, consider the following simple boundary problem: Given a forcing function f, find a solution u such that

$$\boxed{\begin{array}{l} u'' = f, \\ u(0) = u(1) = 0. \end{array}} \tag{1}$$

A boundary problem is called regular if for every forcing function there exists exactly one solution of the differential equation that satisfies the boundary conditions. The *Green's operator* for a regular problem maps every forcing function to its unique solution. If both the differential equation and the boundary conditions are linear, the Green's operator is a linear map. In the classical setting, it turns out that the Green's operator can be represented in the form of an integral operator

$$Gf(x) = \int_a^b g(x,\xi) f(\xi)\, d\xi$$

with a uniquely determined g, called the Green's function for the regular boundary problem.

Based on an operator approach first presented in [16], a symbolic method for computing the Green's operator of two-point boundary problems with constant coefficients was given in [15]. We describe a *symbolic framework* to treat boundary problems for arbitrary linear ordinary differential equations in [17]. A crucial step in the algorithms is the computation of normal forms using a suitable noncommutative Gröbner basis that reflects the essential interactions between certain basic operators. Gröbner bases were introduced by Buchberger in [2] and [3].

Let us reformulate the boundary problem (1). We write V for the complex vector space $C^\infty[0,1]$ and $D\colon V \to V$ for the usual derivation. The boundary conditions can be expressed in terms of the two linear functionals $L\colon f \mapsto f(0)$ and $R\colon f \mapsto f(1)$. We want to find a solution of the differential equation that is annihilated by any linear combination of these two functionals. Using this notation, we can describe the boundary problem (1) by

$$(D^2, [L, R]), \tag{2}$$

where D^2 is a *surjective linear map* on V and $[L, R]$ is a *subspace of the dual space* V^* generated by L and R.

As a second example consider the following boundary problem for the *wave equation* on the domain $\Omega = \mathbb{R} \times \mathbb{R}_{\geq 0}$, now writing V for $C^\infty(\Omega)$: Given a forcing function $f \in V$, find a solution $u \in V$ such that

$$\boxed{\begin{array}{l} u_{tt} - u_{xx} = f, \\ u(x,0) = u_t(x,0) = 0, \end{array}} \tag{3}$$

where the subscripts x and t denote differentiation with respect to the first and second variable, respectively. Note that we use the term "boundary condition/problem" in the general sense of linear conditions. Usually one refers to the above problem as an initial value problem; for a "genuine" boundary problem see the last example of Section 11. We prefer the term "boundary problem" to the more common expression "boundary value problem" since the latter would suggest that boundary conditions are always point evaluation—which is not true in general as one can see in the examples in Section 11.

In (3) we are concerned with the differential operator $\partial_t^2 - \partial_x^2 \colon V \to V$ and the boundary conditions expressed by the infinite family of linear functionals

$$L_x = u \mapsto u(x, 0),\ M_x = u \mapsto u_t(x, 0) \quad \text{for all } x \in \mathbb{R}.$$

In analogy to the previous example, we denote this boundary problem by

$$(\partial_t^2 - \partial_x^2, [L_x, M_x]_{x \in \mathbb{R}}).$$

Note that if u is annihilated by these functionals, it is also annihilated for example by the functionals $\int_0^x u(\eta, 0)\, d\eta$. More precisely, it vanishes on the *orthogonal closure* (see Section 2) of the subspace generated by the boundary conditions; this is the space denoted by $[\ldots]$ above.

Abstracting from the above examples, we arrive in Section 5 at the *definition of a boundary problem*, as a pair consisting of a surjective linear map and an orthogonally closed subspace of the dual space. Every finite-dimensional vector space of the dual is orthogonally closed, but we need the notion of orthogonal closure to deal with infinite dimensional vector spaces (as in the wave equation above) if we are to remain in an algebraic setting without topological assumptions on the vector space or field.

One motivation for us was that understanding algebraic aspects of boundary problems is important for treating boundary problems for linear differential equations by *symbolic computation*, where one usually considers those manipulations of the operators that are independent of the space they act on. Since the surjective linear map may also be a matrix differential operator, this approach can be extended to boundary problems for systems of linear differential equations.

It would be interesting to investigate and extend our results on factoring boundary problems such that additional *topological assumptions* on the vector spaces and operators are taken into account. For example, it should be possible to use a dual pairing [12] instead of a vector space and its algebraic dual. See Wyler [23] for an approach along these lines that deals with generalized Green's operators.

A boundary problem is called *regular* if the boundary conditions specify a complement of the kernel of the defining linear operator. Computing the Green's operator of a regular boundary problem means determining the right inverse corresponding to the prescribed complement of the kernel of the defining operator. We recall and discuss several results for left and right inverses in Section 4. Going back from a Green's operator to the defining operator can be interpreted as solving a suitably defined dual boundary problem, described in Section 5.

In the boundary problems (2) and (3), there is an obvious factorization of the differential operator. So a *natural question* to ask is if one can find two regular first-order boundary problems along this factorization such that the composition of the corresponding Green's operators is the Green's operator for the given problem. Or a similar question: Which boundary problem is solved by the composition of the Green's operators of two given problems?

Answering the second question leads us to the *composition of boundary problems*, discussed in Section 6. Given two boundary problems, we define a new boundary problem as follows: We compose the two linear maps and compute the new boundary conditions by mapping the conditions of the first problem via the transpose of the second linear map and adding them to the boundary conditions of the second problem. See Section 3 for further details on the transpose (in particular, the fact that it maps orthogonally closed subspaces to orthogonally closed subspaces). The composition of two regular boundary problems is again regular, and the corresponding Green's operator is the composition of the Green's operators in reverse order. Analogous results hold for dual boundary problems. Moreover, we show that the solution process can be seen as an anti-isomorphism between the category of regular boundary problems and the category of regular dual boundary problems. For the special case of endomorphisms, we give in Section 8 an interpretation of the composition of boundary problems in terms of a multiplication in a semidirect product of suitably defined monoids and actions.

In Section 7, we characterize all *factorizations of a boundary problem* into two smaller boundary problems. This yields a method for constructing all factorizations of a boundary problem along a given factorization of the defining operator. Factorization of boundary problems is important in the setting of differential equations, where it allows us to split a problem of higher order into subproblems of lower order if we can factor the differential operator. This can be used for solving the given problem by applying symbolic, numeric or hybrid methods to the factor problems. For numerical or hybrid methods one has to consider stability issues [6]: A given well-posed problem should be factored in such a way (if possible) that both lower order problems are still well-posed. For factoring the differential operator, we can exploit algorithms and results about factoring linear ordinary [10, 18, 20, 22] and partial differential operators [8, 9, 21].

For the case of operators with *finite dimensional kernel*, we derive in Sections 9 and 10 additional results, and all the main constructions can be made algorithmic (provided a right inverse of the defining operator is available). This includes in particular the case of boundary problems for ordinary differential equations, which are treated in detail in the forthcoming article [17]. For illustrating the theory developed thus far, we conclude in Section 11 by computing factorizations and Green's operators of the above boundary problems for differential equations.

2 Orthogonally Closed Subspaces

In this section, we summarize the results needed for orthogonally closed subspaces of a vector space and its dual. The notation should remind of the analogous well-known results for Hilbert spaces. See for example Conway [4] and for the Banach space setting Lang [13, pp. 391–394].

First we recall the notion of orthogonality for a bilinear map of modules. Let M and N be left modules over a commutative ring R and

$$b\colon M \times N \to R$$

be a bilinear map. Two vectors $x \in M$ and $y \in N$ are called *orthogonal* with respect to b if $b(x, y) = 0$. We then write $x \perp y$ if the bilinear map b is fixed.

Two subsets $X \subseteq M$ and $Y \subseteq N$ are orthogonal if $x \perp y$ for all $x \in X$ and $y \in Y$. Let $X^{\perp}$ denote the set of all $y \in N$ that are orthogonal to X. This is obviously a submodule of N, which we call the *orthogonal* of X. We define orthogonality on the other side in the same way.

It follows directly from the definition that for any subsets $X_1, X_2 \subseteq M$ we have

$$X_1 \subseteq X_2 \Rightarrow X_1^{\perp} \supseteq X_2^{\perp} \quad \text{and} \quad X_1 \subseteq X_1^{\perp\perp}. \tag{4}$$

These statements hold analogously for subsets of N. Let $\mathbb{P}(M)$ denote the *projective geometry* of a module M, that is, the poset of all submodules (ordered by inclusion). Then the two properties (4) for orthogonality imply that we have an order-reversing Galois connection between the projective geometries $\mathbb{P}(M) \rightleftarrows \mathbb{P}(N)$ defined by

$$M_1 \mapsto M_1^{\perp} \quad \text{and} \quad N_1 \mapsto N_1^{\perp}. \tag{5}$$

Hence we know in particular that $S^{\perp} = S^{\perp\perp\perp}$ for any submodule S of M or N. Moreover, the map $S \mapsto S^{\perp\perp}$ is a closure operator: an extensive ($S \subseteq S^{\perp\perp}$), order-preserving and idempotent self-map. We call a submodule S *orthogonally closed* if $S = S^{\perp\perp}$. The Galois connection restricted to orthogonally closed submodules is an order-reversing bijection. For further details and references on Galois connections we refer to Erné et al. [7].

We now consider the *canonical bilinear form*

$$V \times V^* \to k$$

of a vector space V over a field k and its dual V^* defined by $(v, f) \mapsto f(v)$ and the induced orthogonality on the subspaces. We use the notation $\langle v, f\rangle$ for $f(v)$.

Let $V_1 \subseteq V$ be a subspace. Using the fact that any basis of a subspace can be extended to a basis for V, we see that for any vector $v \in V$ that is not in V_1 there is a linear form $f \in V^*$ with $f(v_1) = 0$ for all $v_1 \in V_1$ and $f(v) = 1$. It follows immediately that every subspace of V is orthogonally closed with respect to the canonical bilinear form $V \times V^* \to k$. Furthermore, we have a natural isomorphism

$$V_1^{\perp} \cong (V/V_1)^*.$$

Indeed, each $f \in V_1^{\perp}$ defines a linear form on V/V_1 since it vanishes on V_1, and it is easy to see that this gives an isomorphism between $V_1^{\perp}$ and $(V/V_1)^*$. This implies in particular that

$$\dim V_1^{\perp} = \operatorname{codim} V_1 \quad \text{if } \operatorname{codim} V_1 < \infty.$$

In the following, we consider subspaces of the dual vector space V^*. We first recall some results for biorthogonal systems. Two families $(v_i)_{i\in I}$ of vectors in V and linear forms $(f_i)_{i\in I}$ in V^* are called *biorthogonal* or said to form a *biorthogonal system* if

$$\langle v_i, f_j\rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

For a biorthogonal system $(v_i)_{i\in I}$ and $(f_i)_{i\in I}$ we can easily compute the coefficients of a linear combination $v = \sum a_i v_i$ with finitely many $a_i \in k$ nonzero. Applying f_j, we obtain

$$\langle v, f_j\rangle = \sum a_i \langle v_i, f_j\rangle = a_j.$$

Evaluating a linear combination $f = \sum a_j f_j$ at v_i gives analogously

$$\langle v_i, f \rangle = \sum a_j \langle v_i, f_j \rangle = a_i.$$

This implies in particular that the v_i and f_i are linearly independent. Moreover, we can easily compute projections onto finite dimensional vector spaces from a biorthogonal system. One can show the following lemma and proposition for finite biorthogonal systems.

Lemma 1 *Let $(v_1, \ldots, v_n) \in V$ and $(f_1, \ldots, f_n) \in V^*$ be biorthogonal. Let $V_1 = [v_1, \ldots, v_n]$ and $\mathcal{F}_1 = [f_1, \ldots, f_n]$ be their linear spans. Then $P \colon V \to V$ defined by*

$$v \mapsto \sum_{i=1}^{n} \langle v, f_i \rangle v_i$$

is a projection with $\operatorname{Im} P = V_1$ and $\operatorname{Ker} P = \mathcal{F}_1^\perp$ so that $V = \mathcal{F}_1^\perp \dot{+} V_1$ and $\operatorname{codim} \mathcal{F}_1^\perp = n$. Moreover, for any $f \in \mathcal{F}_1^{\perp\perp}$ we have

$$f = \sum_{i=1}^{n} \langle v_i, f \rangle f_i,$$

so that $\mathcal{F}_1$ is orthogonally closed.

Proposition 2 *Let $f_1, \ldots, f_n \in V^*$. Then the f_i are linearly independent iff there exist $v_1, \ldots, v_n \in V$ such that (v_i) and (f_i) are biorthogonal.*

We conclude with the previous lemma that every finite dimensional subspace of V^* is orthogonally closed. But if V is infinite dimensional, there are always linear subspaces, and indeed hyperplanes in V^*, that are not orthogonally closed; see for example Köthe [12, p. 71]. Nevertheless, since all subspaces of V are orthogonally closed, we have via the Galois connection (5) an order-reversing bijection between $\mathbb{P}(V)$ and the poset of all orthogonally closed supspaces of V^*, which we denote by $\bar{\mathbb{P}}(V^*)$.

Recall that the projective geometry $\mathbb{P}(V)$ of any vector space V is a complete complemented modular lattice with the join and meet respectively defined as the sum and intersection of subspaces. Modularity means that

$$V_1 + (V_2 \cap V_3) = (V_1 + V_2) \cap V_3 \tag{6}$$

for all $V_1, V_2, V_3 \in \mathbb{P}(V)$ with $V_1 \subseteq V_3$.

Using Equation (4) one can show that $\bar{\mathbb{P}}(V^*)$ is a complete lattice with the meet defined as the intersection and the join defined as the orthogonal closure of the sum of subspaces. Hence the Galois connection (5) is an order-reversing lattice isomorphism between the complete lattices $\mathbb{P}(V)$ and $\bar{\mathbb{P}}(V^*)$. Therefore $\bar{\mathbb{P}}(V^*)$ is also a complemented modular lattice.

Let $V_1, V_2 \in \mathbb{P}(V)$ and $\mathcal{F}_1, \mathcal{F}_2 \in \bar{\mathbb{P}}(V^*)$. Since the meet in $\bar{\mathbb{P}}(V^*)$ is the set-theoretic intersection, we know that

$$(V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp \quad \text{and} \quad (\mathcal{F}_1 \cap \mathcal{F}_2)^\perp = \mathcal{F}_1^\perp + \mathcal{F}_2^\perp. \tag{7}$$

The sum of infinitely many orthogonally closed subspaces is in general not orthogonally closed when V is infinite dimensional. But using the fact that $\bar{\mathbb{P}}(V^*)$ is a modular lattice, one can show the following proposition [12, p. 72].

Proposition 3 *The sum of two orthogonally closed subspaces is orthogonally closed.*

Hence we have also

$$(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp \quad \text{and} \quad (\mathcal{F}_1 + \mathcal{F}_2)^\perp = \mathcal{F}_1^\perp \cap \mathcal{F}_2^\perp. \tag{8}$$

Equations (7) and (8) imply that orthogonality preserves algebraic complements, that is, for direct sums

$$V = V_1 \dot{+} V_2 \quad \text{and} \quad V^* = \mathcal{F}_1 \dot{+} \mathcal{F}_2,$$

we have

$$V^* = V_1^\perp \dot{+} V_2^\perp \quad \text{and} \quad V = \mathcal{F}_1^\perp \dot{+} \mathcal{F}_2^\perp.$$

Every subspace has a complement, hence every orthogonally closed subspace of the dual has an orthogonally closed complement. So if we disregard completeness, the Galois connection (5) is an order-reversing lattice isomorphism between the complemented modular lattices

$$\mathbb{P}(V) \cong \bar{\mathbb{P}}(V^*)$$

with join and meet defined as sum and intersection.

Moreover, for arbitrary (not necessarily orthogonally closed) subspaces $\mathcal{F}_1$ and $\mathcal{F}_2$ of V^* we have

$$\mathcal{F}_1^{\perp\perp} + \mathcal{F}_2^{\perp\perp} = (\mathcal{F}_1 + \mathcal{F}_2)^{\perp\perp}. \tag{9}$$

Using the fact that taking the double orthogonal is a closure operator, we see namely that $\mathcal{F}_1^{\perp\perp} + \mathcal{F}_2^{\perp\perp} \subseteq (\mathcal{F}_1 + \mathcal{F}_2)^{\perp\perp}$; the reverse inclusion follows since the left hand side of (9) is orthogonally closed by Proposition 3. If ${}^{\perp\perp}$ were the closure operator of a topology, Equation (9) would mean that the sum is continuous and closed.

We have already seen that if $\operatorname{codim} V_1 < \infty$ and $\dim \mathcal{F}_1 < \infty$, then

$$\operatorname{codim} V_1 = \dim V_1^\perp \quad \text{and} \quad \dim \mathcal{F}_1 = \operatorname{codim} \mathcal{F}_1^\perp. \tag{10}$$

So we can also consider the restriction of the Galois connection to finite codimensional subspaces of V and finite dimensional subspaces of V^*. This yields an order-reversing lattice isomorphism between modular lattices.

3 The Transpose

Let V and W be vector spaces over a field k and $A\colon V \to W$ a linear map. We recall some basic properties of the *transpose* or *dual* map $A^*\colon W^* \to V^*$ defined by $h \mapsto h \circ A$. Hence

$$\langle Av, h\rangle_W = \langle v, A^*h\rangle_V \quad \text{for all } v \in V,\ h \in W^* \tag{11}$$

with the canonical bilinear forms on W and V, respectively. The map $A \mapsto A^*$ from $L(V,W)$ to $L(W^*,V^*)$ is linear. It is injective since for every nonzero $w \in W$ there exists a linear form $h \in W^*$ with $h(w) \neq 0$. For finite dimensional vector spaces, it is also surjective. We have $(AB)^* = B^*A^*$ for linear maps

$A \in L(U, V)$ and $B \in L(V, W)$. Since ${1_V}^* = 1_{V^*}$, this implies that if A is left (respectively right) invertible, A^* is right (respectively left) invertible, so if A is invertible, also A^* is invertible with $(A^*)^{-1} = (A^{-1})^*$. Moreover, the map $A \mapsto A^*$ is an injective k-algebra anti-homomorphism from $L(V)$ to $L(V^*)$.

In the following, we discuss the relations between the image of subspaces under a linear map, its transpose, and orthogonality. From Equation (11) it follows immediately that the orthogonal of the image of a subspace $V_1 \subseteq V$ is

$$A(V_1)^\perp = (A^*)^{-1}(V_1^\perp). \tag{12}$$

Since $V^\perp = 0$, we have in particular $(\operatorname{Im} A)^\perp = \operatorname{Ker} A^*$. Hence $\operatorname{Ker} A^*$ is orthogonally closed. Taking the orthogonal, we obtain from Equation (12)

$$A(V_1) = (A^*)^{-1}(V_1^\perp)^\perp$$

since every subspace of a vector space is orthogonally closed with respect to the canonical bilinear form. In particular, we have $\operatorname{Im} A = (\operatorname{Ker} A^*)^\perp$. For orthogonally closed subspaces $\mathcal{F}_1 \subseteq V^*$, we obtain

$$A(\mathcal{F}_1^\perp) = (A^*)^{-1}(\mathcal{F}_1)^\perp. \tag{13}$$

Now we consider the images under the transpose. Again we see immediately with Equation (11) that

$$A^*(\mathcal{H}_1)^\perp = A^{-1}(\mathcal{H}_1^\perp) \tag{14}$$

for subspaces $\mathcal{H}_1 \subseteq W^*$. Since $(W^*)^\perp = 0$, we have in particular $(\operatorname{Im} A^*)^\perp = \operatorname{Ker} A$. Taking the orthogonal, we obtain from Equation (14)

$$A^*(\mathcal{H}_1) \subseteq A^*(\mathcal{H}_1)^{\perp\perp} = A^{-1}(\mathcal{H}_1^\perp)^\perp. \tag{15}$$

Note that in general we have a proper inclusion, as one can see by taking the identity map and a subspace that is not orthogonally closed since the right-hand side is orthogonally closed. But we do have equality for orthogonally closed subspaces. In the Banach space setting, identity (17) comes in the context of the Closed Range Theorem [24, p. 205] and holds only for operators with closed range.

Proposition 4 *We have*

$$A^*(W_1^\perp) = A^{-1}(W_1)^\perp \tag{16}$$

for subspaces $W_1 \subseteq W$. In particular,

$$\operatorname{Im} A^* = (\operatorname{Ker} A)^\perp, \tag{17}$$

and the image of A^* is orthogonally closed.

Proof. With Equation (15) and the fact that every subspace a vector space is orthogonally closed with respect to the canonical bilinear form, we know the inclusion $\subseteq$. Conversely, let $f \in A^{-1}(W_1)^\perp$. Then

$$f(v_1) = 0 \quad \text{for all } v_1 \in V \text{ such that } Av_1 \in W_1.$$

So in particular $f(\operatorname{Ker} A) = 0$. We have to find a $h \in W_1^\perp$ such that $f = A^*h$. We define $\tilde{h}\colon \operatorname{Im} A \to K$ by

$$\tilde{h}(Av) = f(v).$$

Then $\tilde{h}$ is well-defined. If $Av_1 = Av_2$, then $v_1 - v_2 \in \operatorname{Ker} A$. Hence $f(v_1) = f(v_2)$ since $f(\operatorname{Ker} A) = 0$. Moreover, note that

$$\tilde{h}(\operatorname{Im} A \cap W_1) = 0.$$

We have to extend $\tilde{h}$ to a linear map $h\colon W \to K$ such that h vanishes on W_1. To this end, let $\tilde{I}_1$ and $\tilde{W}_1$ be complements of $\operatorname{Im} A \cap W_1$ in $\operatorname{Im} A$ and W_1, respectively, so that

$$\operatorname{Im} A = (\operatorname{Im} A \cap W_1) \dotplus \tilde{I}_1 \quad \text{and} \quad W_1 = (\operatorname{Im} A \cap W_1) \dotplus \tilde{W}_1.$$

Then one sees that we have a direct sum

$$\operatorname{Im} A + W_1 = (\operatorname{Im} A \cap W_1) \dotplus \tilde{I}_1 \dotplus \tilde{W}_1.$$

Let $P\colon \operatorname{Im} A + W_1 \to \operatorname{Im} A$ defined by

$$P(\bar{w} + \tilde{w}_1) = \bar{w} \quad \text{where } \bar{w} \in \operatorname{Im} A \text{ and } \tilde{w}_1 \in \tilde{W}_1.$$

Then P is a linear map with $\operatorname{Ker} P = \tilde{W}_1$. We set $h = \tilde{h} \circ P$. Then h is defined on $\operatorname{Im} A + W_1$. We extend h arbitrarily to a linear form on W and denote it again by h. By definition $h = \tilde{h}$ on $\operatorname{Im} A$, and so $f = A^*h$. We have to verify that $h \in W_1^\perp$. Let $w_1 \in W_1$ and

$$w_1 = \bar{w}_1 + \tilde{w}_1 \quad \text{with } \bar{w}_1 \in \operatorname{Im} A \cap W_1 \text{ and } \tilde{w}_1 \in \tilde{W}_1.$$

Then

$$h(w_1) = \tilde{h}(Pw_1) = \tilde{h}(\bar{w}_1) = 0,$$

since $\tilde{h}(\operatorname{Im} A \cap W_1) = 0$, and the proposition is proved. □

We know from Section 2 that the Galois connection (5) gives an isomorphism between $\mathbb{P}(W)$ and the orthogonally closed subspaces $\bar{\mathbb{P}}(W^*)$. So the previous proposition implies

$$A^*(\mathcal{H}_1) = A^{-1}(\mathcal{H}_1^\perp)^\perp \tag{18}$$

for orthogonally closed subspaces $\mathcal{H}_1 \subseteq W^*$. Since the right hand side is orthogonally closed, we obtain the following corollary.

Corollary 5 *The transpose gives an an order-preserving map*

$$\begin{aligned} \bar{\mathbb{P}}(W^*) &\to \bar{\mathbb{P}}(V^*) \\ \mathcal{H}_1 &\mapsto A^*(\mathcal{H}_1) \end{aligned}$$

between orthogonally closed subspaces.

Moreover, using Equation (18) and Equation (14), we see that

$$A^*(\mathcal{H}_1^{\perp\perp}) = A^{-1}(\mathcal{H}_1^\perp)^\perp = A^*(\mathcal{H}_1)^{\perp\perp} \tag{19}$$

for an arbitrary subspace $\mathcal{H}_1 \subseteq W^*$, which means that A^* is "closed" and "continuous" in the hypothetical topological interpretation mentioned after (9).

Finally, we sum up all the identities for the image of subspaces of a linear map and its transpose and orthogonality in the following proposition.

Proposition 6 *Let V and W be vector spaces over a field k and $A\colon V \to W$ a linear map. Then we have*

$$A(V_1)^\perp = (A^*)^{-1}(V_1^\perp), \qquad A(\mathcal{F}_1^\perp) = (A^*)^{-1}(\mathcal{F}_1)^\perp,$$
$$A^*(\mathcal{H}_1)^\perp = A^{-1}(\mathcal{H}_1^\perp), \qquad A^*(W_1^\perp) = A^{-1}(W_1)^\perp,$$

for subspaces $V_1 \subseteq V$, $\mathcal{H}_1 \subseteq W^*$, $W_1 \subseteq W$ and orthogonally closed subspaces $\mathcal{F}_1 \subseteq V^*$. In particular, we have

$$(\operatorname{Im} A)^\perp = \operatorname{Ker} A^*, \qquad \operatorname{Im} A = (\operatorname{Ker} A^*)^\perp,$$
$$(\operatorname{Im} A^*)^\perp = \operatorname{Ker} A, \qquad \operatorname{Im} A^* = (\operatorname{Ker} A)^\perp,$$

for the image and kernel of A and A^*.

Proof. See Equations (12), (13), (14), and (16). □

4 Left and Right Inverses

In this section, we recall and discuss some results for left and right inverses of linear maps. Let V and W be vector spaces over a field k. Let $T\colon V \to W$ and $G\colon W \to V$ be linear maps such that

$$TG = 1.$$

Then T is surjective and G injective, respectively, and GT is a projection with

$$\operatorname{Ker} GT = \operatorname{Ker} T \quad \text{and} \quad \operatorname{Im} GT = \operatorname{Im} G, \tag{20}$$

so that

$$V = \operatorname{Ker} T \dotplus \operatorname{Im} G. \tag{21}$$

Conversely, we can begin with a given surjective or injective linear map and a complement of the kernel and image, respectively, and ask if there exists a corresponding right or left inverse. This is a special case of algebraic generalized inverses as in Nashed and Votruba [14]. We discuss the results for both cases.

Let first $T\colon V \to W$ be a surjective linear map with $K = \operatorname{Ker} T$ and I a complement of K in V, so that

$$V = K \dotplus I.$$

Let P be the projection with $\operatorname{Im} P = K$ and $\operatorname{Ker} P = I$. Then by [14, Theorem 1.20] there exists a unique linear map $G\colon W \to V$ with

$$TG = 1, \quad GT = 1 - P, \quad \text{and} \quad GTG = G.$$

Lemma 7 *The equation $GT = 1 - P$ characterizes G uniquely.*

Proof. The third equation above is obviously redundant, and we show that the first follows from the second. We get for $w = Tv$

$$TGw = TGTv = T(v - Pv) = Tv = w,$$

since $\operatorname{Im} P = \operatorname{Ker} T$. So $TG = 1$ since T is surjective. $\square$

We can also say that given a complement I of $K = \operatorname{Ker} T$, there exists a unique right inverse G with $\operatorname{Im} G = I$. So we have a bijection

$$\{I \in \mathbb{P}(V) \mid V = K \dot{+} I\} \cong \{G \in L(W, V) \mid TG = 1\} \tag{22}$$

between the set of complements of K in V and the set of right inverses of T. Moreover, all right inverses can be described in terms of a fixed one.

Lemma 8 *Given any right inverse $\tilde{G}$ of T, the right inverse corresponding to the complement I is given by*

$$G = (1 - P)\tilde{G},$$

where P is the projection with $\operatorname{Im} P = K$ *and* $\operatorname{Ker} P = I$.

Let now $G\colon W \to V$ be an injective linear map with $I = \operatorname{Im} G$ and K a complement of I in V, so that

$$V = K \dot{+} I.$$

Let P be the projection with $\operatorname{Im} P = K$ and $\operatorname{Ker} P = I$. Since $\operatorname{Im}(1 - P) = \operatorname{Ker} P = I$, there exists by [14, Theorem 1.20] a unique linear map $T\colon V \to W$ with

$$GT = 1 - P, \quad TG = 1, \quad \text{and} \quad TGT = T.$$

Lemma 9 *The equation $GT = 1 - P$ characterizes T uniquely.*

Proof. Note first that since G is injective $\operatorname{Ker} T = \operatorname{Ker} GT = \operatorname{Ker}(1 - P) = K$. Therefore

$$TGT = T - TP = T,$$

which is the third equation above, and hence $TG = (TG)^2$ is a projection. We show that $\operatorname{Ker} TG = 0$, and so TG is the identity. Let $TGw = 0$. Then

$$GTGw = (1 - P)Gw = 0,$$

so that $Gw = PGw$. Since $\operatorname{Ker} P = \operatorname{Im} G$, this implies $Gw = 0$, and thus $w = 0$ because G is injective. $\square$

We can also say that given a complement K of $I = \operatorname{Im} G$, there exists a unique left inverse T with $\operatorname{Ker} T = K$. So we have a bijection

$$\{K \in \mathbb{P}(V) \mid V = K \dot{+} I\} \cong \{T \in L(V, W) \mid TG = 1\} \tag{23}$$

between the set of complements of I in V and the set of left inverses of G. Analogously as above one can describe all left inverses in terms of a fixed one.

Lemma 10 *Given any left inverse $\tilde{T}$ of G, the left inverse corresponding to the complement K is given by*

$$T = \tilde{T}(1 - P),$$

where P is the projection with $\operatorname{Im} P = K$ *and* $\operatorname{Ker} P = I$.

Summing up, the bijections (22) and (23) yield with Lemma 7 and 9 the following proposition.

Proposition 11 *We have a bijection*

$$\{(T, I) \mid T\colon V \to W \text{ surjective, } I \in \mathbb{P}(V) \text{ with } V = \operatorname{Ker} T \dot{+} I\}$$
$$\cong \{(K, G) \mid G\colon W \to V \text{ injective, } K \in \mathbb{P}(V) \text{ with } V = K \dot{+} \operatorname{Im} G\}. \quad (24)$$

Given respectively (T, I) or (K, G), we obtain G or T with $TG = 1$ as the unique solution of

$$GT = 1 - P,$$

where P is the projection with

$$\operatorname{Im} P = \operatorname{Ker} T,\ \operatorname{Ker} P = I \quad \text{and} \quad \operatorname{Im} P = K,\ \operatorname{Ker} P = \operatorname{Im} G,$$

respectively.

The following two propositions describe the inverse image of a composition of an arbitrary and respectively a surjective or injective linear map in terms of one of its right or left inverses.

Proposition 12 *Let U, V, W be vector spaces over a field k. Let $A \in L(V, W)$ be arbitrary, $T \in L(U, V)$ surjective, G a right inverse of T, and $W_1 \subseteq W$ a subspace. Then we have*

$$(AT)^{-1}(W_1) = G(A^{-1}(W_1)) \dot{+} \operatorname{Ker} T$$

for the inverse image of the composite. In particular, we have

$$\operatorname{Ker} AT = G(\operatorname{Ker} A) \dot{+} \operatorname{Ker} T \quad (25)$$

for the kernel of the composite and

$$T^{-1}(V_1) = G(V_1) \dot{+} \operatorname{Ker} T \quad (26)$$

for the inverse image.

Proof. One inclusion is obvious, since

$$AT(G(A^{-1}(W_1)) + \operatorname{Ker} T) = A(A^{-1}(W_1)) + 0 \subseteq W_1.$$

Conversely, let $u \in (AT)^{-1}(W_1)$. Then $Tu = v$ with $v \in A^{-1}(W_1)$. Hence

$$T(u - Gv) = Tu - v = 0$$

and therefore $u \in G(A^{-1}(W_1))+\operatorname{Ker}(T)$. The sum is direct by Equation (21). □

Proposition 13 *Let U, V, W be vector spaces over a field k. Let $A \in L(V, W)$ be arbitrary, $G \in L(U, V)$ injective, T a left inverse of G, and $W_1 \subseteq W$ a subspace. Then we have*

$$(AG)^{-1}(W_1) = T(A^{-1}(W_1) \cap \operatorname{Im} G)$$

for the inverse image of the composite. In particular, we have

$$\operatorname{Ker} AG = T(\operatorname{Ker} A \cap \operatorname{Im} G) \tag{27}$$

for the kernel of the composite and

$$G^{-1}(V_1) = T(V_1 \cap \operatorname{Im} G) \tag{28}$$

for the inverse image.

Proof. Let $v \in A^{-1}(W_1) \cap \operatorname{Im} G$. Since GT is a projection with $\operatorname{Im} GT = \operatorname{Im} G$, see Equation (20), we get

$$AGTv = Av \in W_1,$$

and one inclusion is proved.

Conversely, let $u \in (AG)^{-1}(W_1)$. Then $Gu = v$ with $v \in A^{-1}(W_1) \cap \operatorname{Im} G$. Hence $TGu = u = Tv$, and therefore $u \in T(A^{-1}(W_1) \cap \operatorname{Im} G)$. □

Observe that for $\dim U = \dim V < \infty$, surjectivity as well as injectivity are of course equivalent to bijectivity, and the propositions are trivial. In particular, if T or G is an endomorphism, the propositions are nontrivial only for an infinite dimensional vector space.

5 Boundary Problems and Green's Operators

A *boundary problem* is given by a pair $(T, \mathcal{F})$, where $T\colon V \to W$ is a surjective linear map and $\mathcal{F} \subseteq V^*$ an orthogonally closed subspace of *boundary conditions*. We say that $u \in V$ is a solution of $(T, \mathcal{F})$ for a given $w \in W$, if

$$Tu = w \quad \text{and} \quad f(u) = 0 \quad \text{for all } f \in \mathcal{F}$$

or equivalently $u \in \mathcal{F}^\perp$. A boundary problem $(T, \mathcal{F})$ is *regular* if $\mathcal{F}^\perp$ is a complement of $K = \operatorname{Ker} T$ so that

$$V = K \dotplus \mathcal{F}^\perp.$$

From the previous section we know that then there exists a unique right inverse $G\colon W \to V$ of T with $\operatorname{Im} G = \mathcal{F}^\perp$. We call G the *Green's operator* for the boundary problem $(T, \mathcal{F})$. Since

$$TGw = w \quad \text{and} \quad Gw \in \mathcal{F}^\perp,$$

we see that the Green's operator maps every right-hand side $w \in W$ to its unique solution $u = Gw \in V$. Hence we say that G solves the boundary problem $(T, \mathcal{F})$, and we use the notation

$$G = (T, \mathcal{F})^{-1}.$$

Conversely, if there exists a right inverse G of T for a boundary problem $(T, \mathcal{F})$ such that $\operatorname{Im} G = \mathcal{F}^\perp$, it is regular by Equation (21). Since orthogonality preserves direct sums, we see that $(T, \mathcal{F})$ is regular iff

$$V^* = \mathcal{F} \dotplus K^\perp. \tag{29}$$

By Proposition 6, we have

$$\operatorname{Ker} G^* = (\operatorname{Im} G)^\perp = \mathcal{F}^{\perp\perp} = \mathcal{F} \quad \text{and} \quad \operatorname{Im} T^* = (\operatorname{Ker} T)^\perp = K^\perp \tag{30}$$

for a regular boundary problem $(T, \mathcal{F})$. Given any right inverse $\tilde{G}$ of T, we know with Lemma 8 that the Green's operator for a regular boundary problem $(T, \mathcal{F})$ is given by

$$G = (1 - P)\tilde{G}, \tag{31}$$

where P is the projection with $\operatorname{Im} P = K$ and $\operatorname{Ker} P = \mathcal{F}^\perp$.

If T is invertible, then $(T, 0)$ is the only regular boundary problem for T, and its Green's operator is $(T, 0)^{-1} = T^{-1}$. In particular, we have

$$(1, 0)^{-1} = 1 \tag{32}$$

for the identity operator.

A *dual boundary problem* is given by a a pair (K, G), where $G\colon W \to V$ is an injective linear map and $K \subseteq V$ a subspace of *dual boundary conditions.* We say that $g \in V^*$ is a solution of (K, G) for a given $h \in W^*$ if

$$G^* g = h \quad \text{and} \quad g(v) = 0 \quad \text{for all } v \in K$$

or equivalently $g \in K^\perp$. A dual boundary problem (K, G) is *regular* if K is a complement of $I = \operatorname{Im} G$ so that

$$V = K \dotplus I.$$

From the previous section we know that then there exists a unique left inverse $T\colon V \to W$ of G with $\operatorname{Ker} T = K$. We call T the *dual Green's operator* for the dual boundary problem (K, G). Since $G^* T^* = 1$ and $\operatorname{Im} T^* = K^\perp$ by Proposition 6, we see that

$$G^* T^* h = h \quad \text{and} \quad T^* h \in K^\perp,$$

and so T^* maps every right-hand side $h \in W^*$ to its unique solution $g = T^* h$. Hence we say that T solves the dual boundary problem (K, G), and we use the notation

$$T = (K, G)^{-1}.$$

Conversely, if there exists a left inverse T of G for a dual boundary problem (K, G) such that $\operatorname{Ker} T = K$, it is regular by Equation (21). Given any left inverse $\tilde{T}$ of G, we know with Lemma 10 that the dual Green's operator for a regular dual boundary problem (K, G) is given by

$$T = \tilde{T}(1 - P),$$

where P is the projection with $\operatorname{Im} P = K$ and $\operatorname{Ker} P = I$.

If G is invertible, then $(0, G)$ is the only regular dual boundary problem with G and its dual Green's operator is $(0, G)^{-1} = G^{-1}$. In particular, we have

$$(0, 1)^{-1} = 1 \tag{33}$$

for the identity operator.

For fixed vector spaces V and W we denote the set of all regular (dual) boundary problems respectively by

$$R = \{(T, \mathcal{F}) \mid T\colon V \to W,\ (T, \mathcal{F}) \text{ regular}\}$$

and

$$R^* = \{(K, G) \mid G\colon W \to V,\ (K, G) \text{ regular}\}.$$

We can interpret the bijection (24) between left and right inverses in terms of boundary and dual boundary problems. The main part is always solving a (dual) regular boundary problem, that is, computing its (dual) Green's operator. Note that for boundary problem we specify a complement of the kernel by an orthogonally closed subspace of the dual space.

Proposition 14 *The map*

$$\begin{aligned} R &\to R^* \\ (T, \mathcal{F}) &\mapsto (\operatorname{Ker} T, (T, \mathcal{F})^{-1}) \end{aligned}$$

is a bijection between the sets of regular (dual) boundary problems, and

$$\begin{aligned} R^* &\to R \\ (K, G) &\mapsto ((K, G)^{-1}, (\operatorname{Im} G)^{\perp}). \end{aligned}$$

is its inverse.

Proof. Clear with Proposition 11. □

6 Composing Boundary Problems

Let $(T_1, \mathcal{F}_1)$ and $(T_2, \mathcal{F}_2)$ be boundary problems with

$$T_1\colon V \to W \quad \text{and} \quad T_2\colon U \to V.$$

We define the *composition* of $(T_1, \mathcal{F}_1)$ and $(T_2, \mathcal{F}_2)$ by

$$(T_1, \mathcal{F}_1) \circ (T_2, \mathcal{F}_2) = (T_1 T_2, T_2^*(\mathcal{F}_1) + \mathcal{F}_2). \tag{34}$$

Proposition 15 *The composition of two boundary problems is again a boundary problem.*

Proof. The composition of two surjective maps is surjective. We have to show that $T_2^*(\mathcal{F}_1) + \mathcal{F}_2$ is an orthogonally closed subspace of U^*. But from Corollary 5 we know that $T_2^*(\mathcal{F}_1) \in \bar{\mathbb{P}}(U^*)$ and from Proposition 3 that the sum of two orthogonally closed subspaces is orthogonally closed. □

The composition of boundary problems is associative. Moreover, we have

$$(1_V, 0) \circ (T, \mathcal{F}) = (T, \mathcal{F}) \quad \text{and} \quad (T, \mathcal{F}) \circ (1_W, 0) = (T, \mathcal{F})$$

with $T\colon V \to W$ and 0 the zero-dimensional vector space. So all boundary problems of vector spaces over a fixed field form a category with objects the vector spaces and morphisms the boundary problems.

The next proposition tells us that the composition of boundary problems preserves regularity, and the corresponding Green's operator is the composition of Green's operators in reverse order. Hence the regular boundary problems form a subcategory of the category of all boundary problems. We denote the *category of regular boundary problems* by $\mathcal{R}$.

Proposition 16 *Let $(T_1, \mathcal{F}_1)$ and $(T_2, \mathcal{F}_2)$ be regular boundary problems with Green's operators G_1 and G_2. Then the composition*

$$(T_1, \mathcal{F}_1) \circ (T_2, \mathcal{F}_2) = (T, \mathcal{F})$$

is regular with Green's operator $G_2 G_1$ so that

$$((T_1, \mathcal{F}_1) \circ (T_2, \mathcal{F}_2))^{-1} = (T_2, \mathcal{F}_2)^{-1} \circ (T_1, \mathcal{F}_1)^{-1}.$$

Moreover, the sum

$$\mathcal{F} = T_2^*(\mathcal{F}_1) \dotplus \mathcal{F}_2 \tag{35}$$

is direct.

Proof. We have

$$T_1 T_2 G_2 G_1 = T_1 1 G_1 = T_1 G_1 = 1$$

so that $G_2 G_1$ is a right inverse of $T_1 T_2$. Since $\operatorname{Ker} G_1^* = \mathcal{F}_1$ and $\operatorname{Ker} G_2^* = \mathcal{F}_2$ by Equation (30), we have with Proposition 6 and Equation (25)

$$(\operatorname{Im} G_2 G_1)^\perp = \operatorname{Ker} (G_2 G_1)^* = \operatorname{Ker} G_1^* G_2^* = T_2^*(\mathcal{F}_1) \dotplus \mathcal{F}_2.$$

The proposition now follows by the characterization of regular boundary problems through Green's operators. $\square$

Note that with Equations (19) and (9) we see that

$$T_2^*(\mathcal{F}_1^{\perp\perp}) + \mathcal{F}_2^{\perp\perp} = (T_2^*(\mathcal{F}_1) + \mathcal{F}_2)^{\perp\perp}$$

for arbitrary (not necessarily orthogonally closed) subspaces $\mathcal{F}_1$ and $\mathcal{F}_2$. If the boundary conditions are given by the orthogonal closure of arbitrary subspaces $\mathcal{F}_1$ and $\mathcal{F}_2$, the composition of two boundary problems is equal to

$$(T_1, \mathcal{F}_1^{\perp\perp}) \circ (T_2, \mathcal{F}_2^{\perp\perp}) = (T_1 T_2, (T_2^*(\mathcal{F}_1) + \mathcal{F}_2)^{\perp\perp}). \tag{36}$$

We will use this observation for boundary problems with partial differential equations in Section 11.

Let now (K_2, G_2) and (K_1, G_1) be dual boundary problems with

$$G_2 \colon V \to U \quad \text{and} \quad G_1 \colon W \to V.$$

We define the *composition* of (K_2, G_2) and (K_1, G_1) by

$$(K_2, G_2) \circ (K_1, G_1) = (K_2 + G_2(K_1), G_2 G_1). \tag{37}$$

Obviously, the composition is again a dual boundary problem. It is associative, and we have

$$(0, 1_W) \circ (K, G) = (K, G) \quad \text{and} \quad (K, G) \circ (0, 1_V) = (K, G)$$

with $G\colon W \to V$. So all dual boundary problems of vector spaces over a fixed field form a category.

As we will see, also for dual boundary problems the composition of two regular problems is again regular. Hence the regular dual boundary problems form a subcategory of the category of all dual boundary problems. We denote the *category of regular dual boundary problems* by $\mathcal{R}^*$.

Proposition 17 *Let (K_2, G_2) and (K_1, G_1) be regular dual boundary problems with dual Green's operators T_2 and T_1. Then the composition*

$$(K_2, G_2) \circ (K_1, G_1) = (K, G)$$

is regular with dual Green's operator $T_1 T_2$ so that

$$((K_2, G_2) \circ (K_1, G_1))^{-1} = (K_1, G_1)^{-1} \circ (K_2, G_2)^{-1}.$$

Moreover, the sum $K = K_2 \dotplus G_2(K_1)$ is direct.

Proof. We have

$$T_1 T_2 G_2 G_1 = T_1 1 G_1 = T_1 G_1 = 1$$

so that $T_1 T_2$ is a left inverse of $G_2 G_1$. By Equation (25), we have

$$\operatorname{Ker}(T_1 T_2) = G_2(K_1) \dotplus K_2$$

with $K_1 = \operatorname{Ker} T_1$ and $K_2 = \operatorname{Ker} T_2$. The proposition follows now by the characterization of regular dual boundary problems through dual Green's operators. □

Summing up, we see that solving regular (dual) boundary problems gives an anti-isomorphism between the categories of regular (dual) boundary problems, justifying our terminology for dual boundary problems.

Theorem 18 *The contravariant functor*

$$\begin{aligned} F\colon \mathcal{R} &\to \mathcal{R}^* \\ (T, \mathcal{F}) &\mapsto (\operatorname{Ker} T, (T, \mathcal{F})^{-1}) \end{aligned}$$

is an anti-isomorphism between the categories of regular (dual) boundary problems, and

$$\begin{aligned} F^*\colon \mathcal{R}^* &\to \mathcal{R} \\ (K, G) &\mapsto ((K, G)^{-1}, (\operatorname{Im} G)^{\perp}). \end{aligned}$$

is its inverse.

Proof. By Equation (32) and (33), we have $F(1) = 1$ as well as $F^*(1) = 1$. Hence F and F^* are contravariant functors by Proposition 16 and 17. Finally, $FF^* = 1$ and $F^*F = 1$ by Proposition 14. □

7 Factoring Boundary Problems

Let $(T,\mathcal{F})$ be a boundary problem with $T\colon U \to W$ and assume that we have a factorization

$$(T_1,\mathcal{F}_1)\circ(T_2,\mathcal{F}_2)=(T,\mathcal{F}) \tag{38}$$

into boundary problems with $T_1\colon V\to W$ and $T_2\colon U\to V$. By definition (34), this means that we have a factorization

$$T=T_1T_2$$

for the defining operators and a sum

$$\mathcal{F}=T_2^*(\mathcal{F}_1)+\mathcal{F}_2$$

for the boundary conditions. In this section, we characterize all possible factorizations of a boundary problem into two boundary problems. In particular, we show that if $(T,\mathcal{F})$ is regular, there exists a unique regular left factor $(T_1,\mathcal{F}_1)$, and we describe all right factors $(T_2,\mathcal{F}_2)$.

Given a factorization $T=T_1T_2$ with surjective linear maps T_1 and T_2, we construct all corresponding factorizations into (regular) boundary problems. The boundary conditions for the factor problems can be described in terms of the boundary conditions $\mathcal{F}$ and the factorization $T=T_1T_2$. More precisely, we need $K_2=\operatorname{Ker} T_2$ and an arbitrary right inverse of T_2, which we denote in this section by H_2. We begin without any assumption on the regularity.

Lemma 19 *Let $(T_1,\mathcal{F}_1)\circ(T_2,\mathcal{F}_2)=(T,\mathcal{F})$. Then*

$$T_2^*(\mathcal{F}_1)\subseteq \mathcal{F}\cap K_2^\perp \tag{39}$$

and

$$T_2^*H_2^*(\tilde{\mathcal{F}}_1)=\tilde{\mathcal{F}}_1 \tag{40}$$

for any $\tilde{\mathcal{F}}_1\subseteq K_2^\perp$.

Proof. Note that $\operatorname{Im} T_2^*=K_2^\perp$ by Proposition 6 and $T_2^*(\mathcal{F}_1)\subseteq T_2^*(\mathcal{F}_1)+\mathcal{F}_2=\mathcal{F}$. For the second equation observe that $T_2^*H_2^*$ is a projection with $\operatorname{Im} T_2^*H_2^*=\operatorname{Im} T_2^*=K_2^\perp$ by Equation (20). $\square$

Proposition 20 *Let $T=T_1T_2$ be a factorization with surjective linear maps T_1 and T_2. Let*

$$\tilde{\mathcal{F}}_1\subseteq \mathcal{F}\cap K_2^\perp \quad \textit{and} \quad \mathcal{F}_2\subseteq\mathcal{F}$$

be orthogonally closed subspaces such that $\mathcal{F}=\tilde{\mathcal{F}}_1+\mathcal{F}_2$, and $\mathcal{F}_1=H_2^*(\tilde{\mathcal{F}}_1)$. Then

$$(T_1,\mathcal{F}_1)\circ(T_2,\mathcal{F}_2)=(T,\mathcal{F})$$

is a factorization of $(T,\mathcal{F})$.

Proof. By Corollary 5, we know that $\mathcal{F}_1=H_2^*(\tilde{\mathcal{F}}_1)$ is orthogonally closed, and so $(T_1,\mathcal{F}_1)$ is a boundary problem. Using (40), we observe

$$(T_1,\mathcal{F}_1)\circ(T_2,\mathcal{F}_2)=(T_1T_2,T_2^*H_2^*(\tilde{\mathcal{F}}_1)+\mathcal{F}_2)=(T,\tilde{\mathcal{F}}_1+\mathcal{F}_2)=(T,\mathcal{F}),$$

and the proposition is proved. $\square$

Let now $(T, \mathcal{F})$ be regular with Green's operator G, and assume that we have a factorization $T = T_1 T_2$ with T_1 and T_2 surjective. Then $T_2 G$ is a right inverse of T_1 since

$$T_1 T_2 G = TG = 1.$$

So $(T_1, (\operatorname{Im} T_2 G)^\perp)$ is a regular boundary problem. We can describe its boundary conditions without using G only in terms of $\mathcal{F}$ and T_2 with a right inverse H_2.

Lemma 21 *Let $(T, \mathcal{F})$ be regular with Green's operator G and let $T = T_1 T_2$ be a factorization with surjective linear maps T_1 and T_2. Then*

$$(\operatorname{Im} T_2 G)^\perp = H_2^*(\mathcal{F} \cap K_2^\perp),$$

and $(T_1, H_2^*(\mathcal{F} \cap K_2^\perp))$ is regular with Green's operator $T_2 G$.

Proof. Using Proposition 6 and Equation (27), we obtain

$$(\operatorname{Im} T_2 G)^\perp = \operatorname{Ker} (T_2 G)^* = \operatorname{Ker} G^* T_2^* = H_2^*(\operatorname{Ker} G^* \cap \operatorname{Im} T_2^*).$$

From Equation (30) we know that $\operatorname{Ker} G^* = \mathcal{F}$ and $\operatorname{Im} T_2^* = K_2^\perp$. $\square$

The following theorem tells us that given a regular boundary problem $(T, \mathcal{F})$ and a factorization $T = T_1 T_2$, there is a unique regular left factor described by the previous lemma.

Theorem 22 *Let $(T, \mathcal{F})$ be regular and $T = T_1 T_2$ a factorization with surjective linear maps T_1 and T_2. Then*

$$(T_1, \mathcal{F}_1) \circ (T_2, \mathcal{F}_2) = (T, \mathcal{F})$$

is a factorization with $(T_1, \mathcal{F}_1)$ regular iff

$$\mathcal{F}_1 = H_2^*(\mathcal{F} \cap K_2^\perp)$$

and $\mathcal{F}_2 \subseteq \mathcal{F}$ is an orthogonally closed subspace such that

$$\mathcal{F} = (\mathcal{F} \cap K_2^\perp) + \mathcal{F}_2.$$

Moreover, if $(T_1, \mathcal{F}_1)$ is regular, its Green's operator is $T_2 G$.

Proof. Let $(T_1, \mathcal{F}_1) \circ (T_2, \mathcal{F}_2) = (T, \mathcal{F})$ with $(T, \mathcal{F})$ and $(T_1, \mathcal{F}_1)$ regular. Writing $\bar{\mathcal{F}}_1 = H_2^*(\mathcal{F} \cap K_2^\perp)$, we see with Equation (39) that $\mathcal{F}_1 \subseteq \bar{\mathcal{F}}_1$. Since $(T_1, \mathcal{F}_1)$ is regular by assumption and $(T_1, \bar{\mathcal{F}}_1)$ by the previous lemma, we have

$$\mathcal{F}_1 \dotplus K_1^\perp = \bar{\mathcal{F}}_1 \dotplus K_1^\perp = V^*$$

by Equation (29), so that $\mathcal{F}_1$ and $\bar{\mathcal{F}}_1$ have a common complement. Using modularity, we see that

$$\mathcal{F}_1 = \mathcal{F}_1 + (K_1^\perp \cap \bar{\mathcal{F}}_1) = (\mathcal{F}_1 + K_1^\perp) \cap \bar{\mathcal{F}}_1 = \bar{\mathcal{F}}_1 = H_2^*(\mathcal{F} \cap K_2^\perp).$$

By Equation (40), we have $T_2^*(\mathcal{F}_1) = T_2^* H_2^*(\mathcal{F} \cap K_2^\perp) = \mathcal{F} \cap K_2^\perp$, and so

$$\mathcal{F} = (\mathcal{F} \cap K_2^\perp) + \mathcal{F}_2.$$

Converseley, we know by the previous lemma that $(T_1, H_2^*(\mathcal{F} \cap K_2^\perp))$ is regular, and $(T_1, H_2^*(\mathcal{F} \cap K_2^\perp)) \circ (T_2, \mathcal{F}_2) = (T, \mathcal{F})$ by Proposition 20. $\square$

Finally, assume that *all* boundary problems in the factorization (38) are regular with corresponding Green's operators G, G_1 and G_2. Then we have the factorizations

$$T = T_1 T_2 \quad \text{and} \quad G = G_2 G_1,$$

by Proposition 16, and a direct sum of the boundary conditions

$$\mathcal{F} = T_2^*(\mathcal{F}_1) \dotplus \mathcal{F}_2$$

by Equation (35). Since

$$T_2 G = T_2 G_2 G_1 = G_1,$$

we know from Lemma 21 that $\mathcal{F}_1 = H_2^*(\mathcal{F} \cap K_2^\perp)$. By Equation (40), we obtain $T_2^*(\mathcal{F}_1) = \mathcal{F} \cap K_2^\perp$ so that

$$\mathcal{F} = (\mathcal{F} \cap K_2^\perp) \dotplus \mathcal{F}_2.$$

With the following proposition relating complements, subspaces and orthogonality, we can characterize all regular problems $(T_2, \mathcal{F}_2)$ with $\mathcal{F}_2 \subseteq \mathcal{F}$.

Proposition 23 *Let $K_2 \subseteq K \subseteq V$ be subspaces and $\mathcal{F} \subseteq V^*$ an orthogonally closed subspace such that*

$$V = K \dotplus \mathcal{F}^\perp.$$

Then we have a bijection

$$\{\mathcal{F}_2 \in \bar{\mathbb{P}}(V^*) \mid \mathcal{F}_2 \subseteq \mathcal{F} \wedge V = K_2 \dotplus \mathcal{F}_2^\perp\} \cong \{V_2 \in \mathbb{P}(V) \mid K = V_2 \dotplus K_2\}$$

given by

$$\mathcal{F}_2 \mapsto \mathcal{F}_2^\perp \cap K \quad \textit{and} \quad V_2 \mapsto \mathcal{F} \cap V_2^\perp. \tag{41}$$

Moreover,

$$V = K_2 \dotplus \mathcal{F}_2^\perp \quad \textit{iff} \quad \mathcal{F} = (\mathcal{F} \cap K_2^\perp) \dotplus \mathcal{F}_2,$$

for orthogonally closed subspaces $\mathcal{F}_2 \subseteq \mathcal{F}$.

Proof. Let $\mathcal{F}_2 \subseteq \mathcal{F}$ be orthogonally closed such that $V = K_2 \dotplus \mathcal{F}_2^\perp$. We obtain

$$K = V \cap K = (K_2 + \mathcal{F}_2^\perp) \cap K = K_2 + (\mathcal{F}_2^\perp \cap K),$$

and the sum is direct since $K_2 \cap \mathcal{F}_2^\perp = 0$, so $\mathcal{F}_2^\perp \cap K$ is a complement of K_2 in K. Since $\mathcal{F} \cap K^\perp = 0$, we have

$$\mathcal{F} \cap (\mathcal{F}_2^\perp \cap K)^\perp = \mathcal{F} \cap (\mathcal{F}_2 + K^\perp) = \mathcal{F}_2 + (\mathcal{F} \cap K^\perp) = \mathcal{F}_2.$$

Conversely, let V_2 be a subspace such that $K = V_2 \dotplus K_2$. Since $V = K \dotplus \mathcal{F}^\perp$ and $(\mathcal{F} \cap V_2^\perp)^\perp = \mathcal{F}^\perp + V_2$, we have

$$V = K + \mathcal{F}^\perp = K_2 \dotplus (\mathcal{F}^\perp + V_2) = K_2 \dotplus (\mathcal{F} \cap V_2^\perp)^\perp.$$

Moreover, note that

$$(\mathcal{F} \cap V_2^\perp)^\perp \cap K = (V_2 + \mathcal{F}^\perp) \cap K = V_2 + (\mathcal{F}^\perp \cap K) = V_2$$

since $\mathcal{F}^\perp \cap K = 0$.

Now let $\mathcal{F}_2 \subseteq \mathcal{F}$ be orthogonally closed such that $V = K_2 \dotplus \mathcal{F}_2^\perp$. Let $V_2 = \mathcal{F}_2^\perp \cap K$. Then we know from above that $K = V_2 \dotplus K_2$, so

$$V = K \dotplus \mathcal{F}^\perp = V_2 \dotplus K_2 \dotplus \mathcal{F}^\perp.$$

Since orthogonality preserves direct sums, we obtain

$$V^* = (\mathcal{F} \cap K_2^\perp) \dotplus V_2^\perp.$$

So we have

$$\mathcal{F} = \mathcal{F} \cap V^* = \mathcal{F} \cap ((\mathcal{F} \cap K_2^\perp) + V_2^\perp) = (\mathcal{F} \cap K_2^\perp) + (\mathcal{F} \cap V_2^\perp),$$

and the sum is direct since $(\mathcal{F} \cap K_2^\perp) \cap V_2^\perp = 0$. Since we also know from above that $\mathcal{F} \cap V_2^\perp = \mathcal{F}_2$, the first part of the equivalence is proved.

Conversely, let $\mathcal{F}_2$ be an orthogonally closed subspace such that

$$\mathcal{F} = (\mathcal{F} \cap K_2^\perp) \dotplus \mathcal{F}_2$$

Then $(\mathcal{F} \cap K_2^\perp) \cap \mathcal{F}_2 = 0$ and hence by passing to the orthogonal

$$V = K_2 + \mathcal{F}^\perp + \mathcal{F}_2^\perp = K_2 + \mathcal{F}_2^\perp,$$

the latter since $\mathcal{F}_2^\perp \supseteq \mathcal{F}^\perp$. Moreover, note that

$$\mathcal{F}^\perp = (\mathcal{F} \cap K_2^\perp)^\perp \cap \mathcal{F}_2^\perp = (\mathcal{F}^\perp + K_2) \cap \mathcal{F}_2^\perp = \mathcal{F}^\perp + (K_2 \cap \mathcal{F}_2^\perp).$$

Since $K \cap \mathcal{F}^\perp = 0$, we obtain

$$0 = K \cap (\mathcal{F}^\perp + (K_2 \cap \mathcal{F}_2^\perp)) = (K \cap \mathcal{F}^\perp) + (K_2 \cap \mathcal{F}_2^\perp) = K_2 \cap \mathcal{F}_2^\perp.$$

Hence $V = K_2 \dotplus \mathcal{F}_2^\perp$, and the proposition is proved. □

Corollary 24 *Let $(T, \mathcal{F})$ be regular and T_2 surjective with* $\operatorname{Ker} T_2 \subseteq \operatorname{Ker} T$. *Then* (41) *defines a bijection between*

$$\{\mathcal{F}_2 \subseteq \mathcal{F} \mid (T_2, \mathcal{F}_2) \text{ regular}\}$$

and complements of $\operatorname{Ker} T_2$ *in* $\operatorname{Ker} T$. *Moreover, $(T_2, \mathcal{F}_2)$ is regular iff $\mathcal{F}_2$ is an orthogonally closed complement of $(\mathcal{F} \cap K_2^\perp)$ in $\mathcal{F}$.*

The following corollary allows us to compute the boundary conditions for the unique regular left factor if we have the Green's operator for a regular right factor.

Corollary 25 *Let $(T, \mathcal{F})$ be regular and T_2 surjective with* $\operatorname{Ker} T_2 \subseteq \operatorname{Ker} T$. *Then*

$$G_2^*(\mathcal{F}) = G_2^*(\mathcal{F} \cap K_2^\perp)$$

if G_2 is the Green's operator for $(T_2, \mathcal{F}_2)$ regular with $\mathcal{F}_2 \subseteq \mathcal{F}$.

Proof. If $G_2 = (T_2, \mathcal{F}_2)^{-1}$ with $\mathcal{F}_2 \subseteq \mathcal{F}$, then

$$\mathcal{F} = (\mathcal{F} \cap K_2^\perp) \dotplus \mathcal{F}_2,$$

by the previous corollary. Since $\operatorname{Ker} G_2^* = \mathcal{F}_2$ by Equation (30), this implies $G_2^*(\mathcal{F}) = G_2^*(\mathcal{F} \cap K_2^\perp)$. □

Summing up, we can now characterize and construct all possible factorizations of a regular boundary problem into two regular boundary problems given a factorization of the defining operator.

Theorem 26 *Let $(T, \mathcal{F})$ be regular and $T = T_1 T_2$ a factorization with surjective linear maps T_1 and T_2. Then*

$$(T_1, \mathcal{F}_1) \circ (T_2, \mathcal{F}_2) = (T, \mathcal{F})$$

is a factorization with $(T_2, \mathcal{F}_2)$ regular iff

$$\mathcal{F}_1 = H_2^*(\mathcal{F} \cap K_2^\perp)$$

and $\mathcal{F}_2 \subseteq \mathcal{F}$ is an orthogonally closed subspace such that

$$\mathcal{F} = (\mathcal{F} \cap K_2^\perp) \dotplus \mathcal{F}_2.$$

In particular, the left factor $(T_1, \mathcal{F}_1)$ is necessarily regular.

Proof. Let $(T_1, \mathcal{F}_1) \circ (T_2, \mathcal{F}_2) = (T, \mathcal{F})$ with $(T, \mathcal{F})$ and $(T_2, \mathcal{F}_2)$ regular. Let G_2 be the Green's operator for $(T_2, \mathcal{F}_2)$. Since $\operatorname{Ker} G_2^* = \mathcal{F}_2$ by Equation (30) and $\mathcal{F} = T_2^*(\mathcal{F}_1) + \mathcal{F}_2$, we obtain $G_2^*(\mathcal{F}) = \mathcal{F}_1$. With the previous corollary this yields

$$\mathcal{F}_1 = G_2^*(\mathcal{F} \cap K_2^\perp),$$

and so $(T_1, \mathcal{F}_1)$ is regular by Lemma 21. The theorem follows with Corollary 24 and Theorem 22. □

8 A Monoid of Boundary Problems

In this section, we consider boundary problems with *endomorpisms*; this case is also the basis for the symbolic computation treatment in [17]. Having endomorphisms, the composition of boundary problems (34) and dual boundary problems (37) coincides with the multiplication in a reverse semidirect product of suitable defined monoids and actions. Moreover, the contravariant functors from Theorem 18 between regular (dual) boundary problems specialize to anti-isomorphisms between the submonoids of regular (dual) boundary problems.

Given a monoid action, one can define the semidirect product of monoids just as for groups. In contrast to groups, one has to distinguish between left and right actions and accordingly define the multiplication for semidirect products.

We recall the definitions. Let M and N be monoids. Following a convention introduced by Eilenberg [5], which also fits perfectly with our application, we write the product in M additively (without assuming commutativity). Given a left action of N on M, denoted by $n \cdot m$, and specified by a homomorphism

$$\varphi\colon N \to \mathrm{End}M,$$

the *semidirect product* $M \rtimes_\varphi N$ is the set $M \times N$ with the multiplication "from the left"

$$(m_1, n_1)(m_2, n_2) = (m_1 + n_1 \cdot m_2, n_1 n_2) = (m_1 + \varphi_{n_1}(m_2), n_1 n_2).$$

One verifies that this multiplication is associative with identity $(0, 1)$, so the semidirect product $M \rtimes_\varphi N$ is indeed a monoid.

Analogously, given a right action of N on M, denoted by $m \cdot n$, and specified by an anti-homomorphism

$$\varphi\colon N \to \mathrm{End}M,$$

the *reverse semidirect product* $N \ltimes_\varphi M$ is the set $N \times M$ with the multiplication "from the right"

$$(n_1, m_1)(n_2, m_2) = (n_1 n_2, m_1 \cdot n_2 + m_2,) = (n_1 n_2, \varphi_{n_2}(m_1) + m_2).$$

Again $N \ltimes_\varphi M$ is a monoid with identity $(1, 0)$.

Let now V be a vector space and $L(V)$ the monoid of endomorphisms with respect to composition. Considering $\mathbb{P}(V)$ as an additive monoid, $L(V)$ acts on $\mathbb{P}(V)$ from the left by $A \cdot V_1 = A(V_1)$, so we have a homomorphism

$$\varphi\colon L(V) \to \mathrm{End}\mathbb{P}(V) \quad \text{with} \quad \varphi_A(V_1) = A(V_1).$$

The multiplication in the semidirect product $\mathbb{P}(V) \rtimes_\varphi L(V)$ is

$$(V_1, A_1)(V_2, A_2) = (V_1 + A_1(V_2), A_1 A_2),$$

which is exactly the definition (37) of the composition of dual boundary problems. Writing H for the submonoid of all injective endomorphisms, we see that the semidirect product $\mathbb{P}(V) \rtimes_\varphi H$ is the *monoid of dual boundary problems*. The regular dual boundary problems form a submonoid

$$R^* = \{(K, G) \in \mathbb{P}(V) \times H \mid (K, G) \text{ regular}\}$$

since the composition of two regular dual boundary problems is regular by Proposition 17.

We now discuss the situation for boundary problems. By Propostion 3, the sum of two orthogonally closed subspaces is orthogonally closed, so $\bar{\mathbb{P}}(V^*)$ is an additive monoid. We know from Corollary 5 that the transpose maps orthogonally closed subspaces to orthogonally closed subspaces. Hence $L(V)$ acts on $\bar{\mathbb{P}}(V^*)$ from the right via the transpose $\mathcal{F} \cdot A = A^*(\mathcal{F})$, and we have the anti-homomorphism

$$\varphi\colon L(V) \to \mathrm{End}\bar{\mathbb{P}}(V^*) \quad \text{with} \quad \varphi_A(\mathcal{F}) = A^*(\mathcal{F}).$$

The multiplication in the reverse semidirect product $L(V) \ltimes_\varphi \bar{\mathbb{P}}(V^*)$ is

$$(A_1, \mathcal{F}_1)(A_2, \mathcal{F}_2) = (A_1 A_2, A_2^*(\mathcal{F}_1) + \mathcal{F}_2),$$

which is the definition (34) of the composition of boundary problems. Writing S for the submonoid of all surjective endomorphisms, we see that the reverse semidirect product $S \ltimes_\varphi \bar{\mathbb{P}}(V^*)$ is the *monoid of boundary problems.* The regular boundary problems form a submonoid

$$R = \{(T, \mathcal{F}) \in S \times \bar{\mathbb{P}}(V^*) \mid (T, \mathcal{F}) \text{ regular}\}$$

since the composition of two regular boundary problems is regular by Proposition 16.

Solving regular (dual) boundary problems gives an anti-isomorphism between the monoids of regular (dual) boundary problems. More precisely, we have the following result as a special case of Theorem 18.

Proposition 27 *The map*

$$\begin{aligned} R &\to R^* \\ (T, \mathcal{F}) &\mapsto (\operatorname{Ker} T, (T, \mathcal{F})^{-1}) \end{aligned}$$

is an anti-isomorphism between the monoids of regular (dual) boundary problems, and

$$\begin{aligned} R^* &\to R \\ (K, G) &\mapsto ((K, G)^{-1}, (\operatorname{Im} G)^{\perp}). \end{aligned}$$

is its inverse.

Given some submonoid S_1 of all surjective endomorpisms S, we can consider the monoid of boundary problems $S_1 \ltimes \bar{\mathbb{P}}(V^*)$ with linear maps in S_1. We can also restrict the boundary conditions to a submonoid F of $\bar{\mathbb{P}}(V^*)$ if F is closed under S_1 in the sense that

$$T^*(\mathcal{F}) \in F, \quad \text{for all } T \in S_1 \text{ and } \mathcal{F} \in F,$$

so that S_1 acts on F. In all such cases, the regular boundary problems form a submonoid. As an example, take the submonoid of surjective endomorphisms with finite dimensional kernel with finite dimensional subspaces of boundary conditions.

Analogously, we can consider submonoids of all injective endomorphisms and restrict the dual boundary conditions to suitable submonoids of $\mathbb{P}(V)$. The corresponding dual problems for the previous example are injective endomorphisms with finite codimensional image with finite dimensional subspaces as dual boundary conditions.

Note that with the results from Section 7, given a factorization in S_1, we can construct all factorizations of a (regular) boundary problem into (regular) boundary problems with arbitrary boundary conditions. If we restrict the boundary conditions to a submonoid F, we have to check whether the constructed boundary conditions are again in F.

9 Dimension and Codimension

Recall that for subspaces V_1 and V_2 of a vector space V we have

$$\dim(V_1 + V_2) + \dim(V_1 \cap V_2) = \dim V_1 + \dim V_2$$

and analogously for the codimension

$$\operatorname{codim}(V_1 + V_2) + \operatorname{codim}(V_1 \cap V_2) = \operatorname{codim} V_1 + \operatorname{codim} V_2.$$

Note that if V is finite dimensional, the second equation is a consequence from the first and the equation

$$\dim V_1 + \operatorname{codim} V_1 = \dim V.$$

For V finite dimensional, we obtain similarly the equation

$$\operatorname{codim}(V_1 + V_2) + \dim V_1 = \dim(V_1 \cap V_2) + \operatorname{codim} V_2$$

relating the codimension of the sum with the dimension of the intersection of two subspaces. We show that this equation holds for arbitrary vector spaces.

Proposition 28 *We have*

$$\operatorname{codim}(V_1 + V_2) + \dim V_1 = \dim(V_1 \cap V_2) + \operatorname{codim} V_2$$

for subspaces V_1 and V_2 of a vector space V.

Proof. Let $\tilde{V}_1$ and $\tilde{V}_2$ be complements of $V_1 \cap V_2$ in V_1 and V_2, respectively, so that

$$V_1 = \tilde{V}_1 \dotplus (V_1 \cap V_2) \quad \text{and} \quad V_2 = \tilde{V}_2 \dotplus (V_1 \cap V_2).$$

Then one sees that we have a direct sum

$$V_1 + V_2 = \tilde{V}_1 \dotplus \tilde{V}_2 \dotplus (V_1 \cap V_2).$$

Let $\tilde{W}$ be a complement of $V_1 + V_2$ in V so that

$$V = (V_1 + V_2) \dotplus \tilde{W} = \tilde{V}_1 \dotplus \tilde{V}_2 \dotplus (V_1 \cap V_2) \dotplus \tilde{W}.$$

Hence

$$\operatorname{codim}(V_1 + V_2) = \dim \tilde{W} \quad \text{and} \quad \operatorname{codim} V_2 = \dim(\tilde{W} + \tilde{V}_1).$$

Computing the dimension of the subspace

$$\tilde{W} \dotplus \tilde{V}_1 \dotplus (V_1 \cap V_2)$$

in two different ways, we obtain

$$\begin{aligned}\operatorname{codim}(V_1 + V_2) + \dim V_1 &= \dim \tilde{W} + \dim(\tilde{V}_1 + (V_1 \cap V_2)) \\ &= \dim(V_1 \cap V_2) + \dim(\tilde{W} + \tilde{V}_1) = \dim(V_1 \cap V_2) + \operatorname{codim} V_2,\end{aligned}$$

and the proposition is proved. □

If V_1 is finite dimensional and V_2 finite codimensional, all dimensions and codimensions in the above proposition are finite, and we obtain the following corollaries.

Corollary 29 *Let V_1 and V_2 be subspaces of a vector space V with* $\dim V_1 < \infty$ *and* $\operatorname{codim} V_2 < \infty$. *Then*

$$\operatorname{codim}(V_1 + V_2) - \dim(V_1 \cap V_2) = \operatorname{codim} V_2 - \dim V_1.$$

In particular,

$$\dim(V_1 \cap V_2) = \operatorname{codim}(V_1 + V_2) \;\Leftrightarrow\; \dim V_1 = \operatorname{codim} V_2.$$

Corollary 30 *Let V_1 and V_2 be subspaces of a vector space V with* $\dim V_1 < \infty$ *and* $\operatorname{codim} V_2 < \infty$. *Then*

$$V_1 \dot{+} V_2 = V$$

iff

$$V_1 \cap V_2 = 0 \quad \textit{and} \quad \dim V_1 = \operatorname{codim} V_2$$

iff

$$V_1 + V_2 = V \quad \textit{and} \quad \dim V_1 = \operatorname{codim} V_2.$$

So for testing whether two subspaces V_1 and V_2 with $\dim V_1 = \operatorname{codim} V_2 < \infty$ establish a direct decomposition $V = V_1 \dot{+} V_2$, we have to check only one of the two defining conditions $V_1 \cap V_2 = 0$ and $V_1 + V_2 = V$.

The hypothesis that the dimensions are finite is necessary. Let k be a field, $V = k^{\mathbb{N}}$, and consider for example the two subspaces

$$\begin{aligned} V_1 &= \{(0, x_1, 0, x_2, 0, x_3, \ldots) \mid (x_n) \in k^{\mathbb{N}}\} \\ V_2 &= \{(0, 0, x_1, 0, x_2, 0, x_3, \ldots) \mid (x_n) \in k^{\mathbb{N}}\}. \end{aligned}$$

Then $\dim V_1 = \operatorname{codim} V_2 = \dim V = \infty$, $V_1 \cap V_2 = 0$ but $\operatorname{codim}(V_1 + V_2) = 1$.

We use the following corollary in the next section as a regularity test for boundary problems with finite dimensional kernels and boundary conditions.

Corollary 31 *Let $V_1 = [v_1, \ldots, v_m]$ be a subspace of a vector space V and $\mathcal{F}_1 = [f_1, \ldots, f_n]$ a subspace of V^* with f_i and v_j linearly independent. Then*

$$V = V_1 \dot{+} \mathcal{F}_1^{\perp}$$

is a direct sum iff $m = n$ and the matrix $(f_i(v_j))$ is regular.

Proof. By Equation (10), $\operatorname{codim} \mathcal{F}_1^{\perp} = \dim \mathcal{F}_1$, so we know from the previous corollary that

$$V = V_1 \dot{+} \mathcal{F}_1^{\perp}$$

is a direct sum iff $V_1 \cap \mathcal{F}_1^{\perp} = 0$ and $m = n$. Let $B = (f_i(v_j))$ with columns b_j. Now note that B is singular iff there exists a linear combination $\sum \lambda_j b_j = 0$ with at least one $\lambda_j \neq 0$ iff there exists a nonzero $u = \sum \lambda_j v_j$ in $V_1 \cap \mathcal{F}_1^{\perp}$. $\square$

10 Finitely Many Boundary Conditions

In this section, we specialize some results and discuss algorithmic aspects for boundary problems where the corresponding linear maps have finite dimensional kernels and the spaces of boundary conditions are finite dimensional. Note that this includes boundary value problems for (systems of) ordinary differential equations and systems of partial differential equations with finite dimensional solution space.

More precisely, we consider boundary problems $(T, \mathcal{F})$ where $T\colon V \to W$,

$$\dim K < \infty \quad \text{and} \quad \mathcal{F} = [f_1, \dots, f_n]$$

with $K = \operatorname{Ker} T$. We can rewrite the condition that $u \in V$ is a solution of the boundary problem $(T, \mathcal{F})$ for a given $w \in W$ in the following traditional form

$$\boxed{\begin{array}{l} Tu = w, \\ f_1(u), \dots, f_n(u) = 0. \end{array}}$$

By Corollary 31, a necessary condition for the regularity of $(T, \mathcal{F})$ is

$$\dim \operatorname{Ker} T = \dim \mathcal{F},$$

meaning that we have the "correct" number of boundary conditions. Moreover, we get the following algorithmic regularity test for boundary problems (to be be found in Kamke [11, p. 184] for the special case of two-point boundary conditions).

Proposition 32 *A boundary problem $(T, \mathcal{F})$ with $\dim \operatorname{Ker} T = \dim \mathcal{F}$ is regular iff the matrix*

$$\begin{pmatrix} f_1(u_1) & \cdots & f_1(u_n) \\ \vdots & \ddots & \vdots \\ f_n(u_1) & \cdots & f_n(u_n) \end{pmatrix}$$

is regular, where the f_i and u_j are any basis of respectively $\mathcal{F}$ and $\operatorname{Ker} T$.

Let T be a fixed surjective linear map. By Equation (31), given any right inverse $\tilde{G}$ of T, the Green's operator for a regular boundary problem $(T, \mathcal{F})$ is given by

$$G = (1 - P)\tilde{G},$$

where P is the projection with $\operatorname{Im} P = K$ and $\operatorname{Ker} P = \mathcal{F}^\perp$. If T has a finite dimensional kernel with basis $u_1, \dots, u_n$, we can easily describe the projection P in terms of a basis $f_1, \dots, f_n$ of $\mathcal{F}$. Since the matrix $B = (f_i(u_j))$ is regular by the previous proposition, we can define

$$(\tilde{f}_1, \dots, \tilde{f}_n)^t = B^{-1}(f_1, \dots, f_n)^t.$$

Then the $(\tilde{f}_i)$ and (u_j) are biorthogonal, and $P\colon V \to V$ defined by

$$v \mapsto \sum_{i=1}^{n} \langle v, \tilde{f}_i \rangle u_i$$

is the projection with $\operatorname{Im} P = K$ and $\operatorname{Ker} P = \mathcal{F}^\perp$ by Lemma 1.

Given a factorization $T = T_1 T_2$ and a right inverse H_2 of T_2, we know from Theorem 26 how to construct all possible factorizations of a regular boundary problem $(T, \mathcal{F})$ into two regular problems. The boundary conditions for the left factor $(T_1, \mathcal{F}_1)$ are uniquely given by

$$\mathcal{F}_1 = H_2^*(\mathcal{F} \cap K_2^\perp),$$

and all regular boundary problems $(T_2, \mathcal{F}_2)$ correspond to direct sums

$$\mathcal{F} = (\mathcal{F} \cap K_2^\perp) \dotplus \mathcal{F}_2.$$

In the following, we discuss how all such factorizations can be computed by linear algebra if T has a finite dimensional kernel.

Let $(T, \mathcal{F})$ be regular, $K = \operatorname{Ker} T$, $K_2 = \operatorname{Ker} T_2$, and $f_1, \dots, f_{m+n}$ a basis of $\mathcal{F}$. Choose a basis

$$u_1, \dots, u_m, u_{m+1}, \dots, u_{m+n}$$

of K such that $u_1, \dots, u_m$ is basis of K_2, and let

$$B = \begin{pmatrix} f_1(u_1) & \dots & f_1(u_m) & f_1(u_{m+1}) & \dots & f_1(u_{m+n}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{m+n}(u_1) & \dots & f_{m+n}(u_m) & f_{m+n}(u_{m+1}) & \dots & f_{m+n}(u_{m+n}) \end{pmatrix}. \quad (42)$$

Since B is regular, we can perform row operations corresponding to a regular matrix P such that

$$P\,B = \begin{pmatrix} B_2 & C \\ 0 & D \end{pmatrix} \quad (43)$$

is a block matrix, where B_2 is a regular $m \times m$ matrix. Let

$$(\tilde{f}_1, \dots, \tilde{f}_m, \tilde{f}_{m+1}, \dots, \tilde{f}_{m+n})^t = P\,(f_1, \dots, f_{m+n})^t, \quad (44)$$

that is,

$$\tilde{f}_i = \sum_{j=1}^{m+n} P_{ij} f_j,$$

and $\mathcal{F}_2 = [\tilde{f}_1, \dots, \tilde{f}_m]$. Then obviously $[\tilde{f}_{m+1}, \dots, \tilde{f}_{m+n}] \subseteq \mathcal{F} \cap K_2^\perp$ and since $\dim(\mathcal{F} \cap K_2^\perp) = \operatorname{codim}(\mathcal{F}^\perp + K_2) = n$, they are equal. So

$$\mathcal{F} = (\mathcal{F} \cap K_2^\perp) \dotplus \mathcal{F}_2$$

is a direct sum. Conversely, it is clear that any such direct sum given by bases $\mathcal{F}_2 = [\tilde{f}_1, \dots, \tilde{f}_m]$ and $\mathcal{F} \cap K_2^\perp = [\tilde{f}_{m+1}, \dots, \tilde{f}_{m+n}]$ with P as in (44) gives a block matrix as in Equation (43). By Theorem 26, we know that

$$(T, \mathcal{F}) = (T_1, \mathcal{F}_1) \circ (T_2, \mathcal{F}_2)$$

is a factorization into regular boundary problems with

$$\mathcal{F}_1 = [H_2^*(\tilde{f}_{m+1}), \dots, H_2^*(\tilde{f}_{m+n})] \quad \text{and} \quad \mathcal{F}_2 = [\tilde{f}_1, \dots, \tilde{f}_m]. \quad (45)$$

Note that if H_2 is the Green's operator for a regular right factor $(T_2, \mathcal{F}_2)$ with $\mathcal{F}_2 \subseteq \mathcal{F}$, we have $H_2^*(\mathcal{F}) = H_2^*(\mathcal{F} \cap K_2^\perp)$ by Corollary 25. So we can compute the uniquely determined boundary conditions $\mathcal{F}_1$ simply by applying H_2^* to the boundary conditions $\mathcal{F}$; see the examples in the next section.

11 Examples for Differential Equations

Let us now illustrate our algebraic approach to abstract boundary problems in the concrete setting of differential equations, taking up the examples posed in the Introduction.

We want to factor the *two-point boundary problem* $(D^2, [L, R])$ of Equation (1) into two regular problems with $T_1 = T_2 = D$. The indefinite integral $A = \int_0^x$ is the Green's operator for the regular right factor $(D, [L])$. By Corollary 25, the boundary conditions for the unique left factor are given by

$$A^*[L, R] = [0, RA] = [RA],$$

where $RA = \int_0^1$ is the definite integral. So we obtain the factorization

$$(D, [RA]) \circ (D, [L]) = (D^2, [L, R])$$

or

$$\boxed{\begin{array}{l} u' = f \\ \int_0^1 u(\xi)\, d\xi = 0 \end{array}} \circ \boxed{\begin{array}{l} u' = f \\ u(0) = 0 \end{array}} = \boxed{\begin{array}{l} u'' = f \\ u(0) = u(1) = 0 \end{array}}$$

in the notation from the Introduction. Note that the boundary condition for the left factor is an integral condition. Such conditions are not considered in the classical setting of two-point boundary problems but are known in the literature as Stieltjes boundary conditions [1]. We check this factorization by multiplying the two boundary problems according to Definition (34). Note that

$$(D, [RA]) \circ (D, [L]) = (D^2, [D^*(RA), L])$$

and $D^*(RA) = RAD = \int_0^1 D = L - R$ so that

$$[D^*(RA), L] = [L - R, R] = [L, R].$$

To illustrate the method from the previous section, we factor the boundary problem $(D^2, [LD, R])$. We use again the indefinite integral $A = (D, [L])^{-1}$ as a right inverse of D, but for this boundary problem it is not a Green's operator for a regular right factor since $L \notin [LD, R]$. Hence we cannot simply apply A^* to the boundary conditions as we did before since this would give us two conditions

$$A^*[LD, R] = [LDA, RA] = [L, RA]$$

for a first-order problem. So we have to proceed as described in the previous section. A suitable basis for $\operatorname{Ker} D^2$ is $(1, x)$. Evaluating the boundary conditions $[LD, R]$ on $(1, x)$ yields

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

for the matrix B from Equation (42). Swapping the first and the second row gives a block triangular matrix as in Equation (43). So by Equation (45), the boundary condition is given by $A^*(LD) = L$ for the left factor and by R for the right factor, and we obtain the factorization

$$(D, [L]) \circ (D, [R]) = (D^2, [LD, R]).$$

See [17] for a general discussion of solving and factoring boundary problems for ordinary differential equations in an algorithmic context.

As an example of a boundary problem for a partial differential equation, we return to the *wave equation* in (3) from the Introduction. We write it as

$$\mathcal{W} = (\partial_t^2 - \partial_x^2, [u(x,0), u_t(x,0)]),$$

where $u(x,0)$ and $u_t(x,0)$ are short for the functionals $u \mapsto u(x,0)$ and $u \mapsto u_t(x,0)$, respectively, and $[\ldots]$ denotes the orthogonal closure of the subspace generated by these functionals with x ranging over $\mathbb{R}$. The Green's operator of $\mathcal{W}$ is given by

$$Gf(x,t) = \frac{1}{2} \int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} f(\xi,\tau)\, d\xi\, d\tau, \tag{46}$$

as can be found in the literature [19, p. 485]. We show that one can determine G by constructing a factorization of $\mathcal{W}$ along the factorization

$$\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x)(\partial_t + \partial_x).$$

A regular right factor is given by

$$\mathcal{W}_2 = (\partial_t + \partial_x, [u(x,0)]).$$

In general, choosing boundary conditions in such a way that they make up a regular boundary problem for a given first-order right factor of a linear partial differential operator amounts to a geometric problem involving the characteristics. The Green's operator for $\mathcal{W}_2$ can easily be computed as

$$G_2 f(x,t) = \int_{x-t}^{x} f(\xi, \xi - x + t)\, d\xi$$

and can be used for finding the boundary conditions for the uniquely determined left factor

$$\mathcal{W}_1 = (\partial_t - \partial_x, G_2^*[u(x,0), u_t(x,0)]) = (\partial_t - \partial_x, [u(x,0)])$$

by Corollary 25. One can verify the factorization $\mathcal{W} = \mathcal{W}_1\mathcal{W}_2$, taking into account (36). The Green's operator for $\mathcal{W}_1$ is analogously given by

$$G_1 f(x,t) = \int_x^{x+t} f(\xi, x - \xi + t)\, d\xi,$$

and all we have to do now is to compute the composite

$$G_2 G_1 f(x,t) = \int_{x-t}^{x} \int_{\tau}^{2\tau - x + t} f(\xi, 2\tau - \xi - x + t)\, d\xi\, d\tau,$$

which is the Green's operator of $\mathcal{W}$ by Theorem 26. Since G and G_2G_1 solve the same regular boundary problem, we know that $G = G_2G_1$, as one may also verify directly by a change of variables.

The above methodology can also be transferred to the computationally more involved case of the wave equation on the *bounded interval* $[0,1]$, succinctly expressed in our notation by

$$\mathcal{V} = (\partial_t^2 - \partial_x^2, [u(x,0), u_t(x,0), u(0,t), u(1,t)].$$

As indicated above, one can find a factorization $\mathcal{V} = \mathcal{V}_1 \circ \mathcal{V}_2$ with

$$\mathcal{V}_1 = (u_t - u_x, [u(x,0), \textstyle\int_{\max(1-t,0)}^1 u(\xi, \xi + t - 1)\, d\xi]),$$
$$\mathcal{V}_2 = (u_t + u_x, [u(x,0), u(0,t)]).$$

Unlike in the unbounded case, the Green's operator for $\mathcal{V}$ involves a finite sum whose upper bound depends on the argument (x,t). These complication are reflected in the Green's operator of the left factor $\mathcal{V}_1$, whose computation leads to a simple functional equation. A systematic investigation of partial differential equations with integral boundary conditions is a subject of future work.

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