# Solution of Ill-Posed Problems via Adaptive Grid Regularization: Convergence Analysis

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**Abstract.** In the papers [15, 16] a new regularization method, *adaptive grid regularization*, was presented. Numerical results there show in a convincing way that this method is a powerful tool to identify discontinuities of solutions of ill-posed problems. It is the aim of this paper to give a convergence analysis for this new method.

## 1. Introduction

When studying (linear or nonlinear) ill-posed problems

$$F(x) = y\,, \qquad F : \mathcal{D}(F)(\subset \mathcal{X}) \to \mathcal{Y}\,, \tag{1.1}$$

where usually only noisy measurements $y^\delta$ of $y$ with $\|y^\delta - y\| \le \delta$ are known, where $\mathcal{Y}$ is a Hilbert space and $\mathcal{X}$ is a Banach space of functions defined on $\Omega$, an open bounded convex subset of $\mathbb{R}^d$ $(d = 1, 2, 3)$ with Lipschitz boundary, it is well known by now that standard regularization methods (cf., e.g., [7, 8]) are not appropriate for ill-posed problems with discontinuous solutions, since they have a smoothing effect on regularized solutions.

If one expects discontinuous solutions, special care has to be taken in choosing the regularization method. Bounded variation regularization has turned out to be an effective method [1, 4, 18] when dealing with such problems. In [17] a new approach for regularizing problems with discontinuous solutions was introduced, regularization for curve representations. The essence of this method is to replace a discontinuous function by its continuous graph and to apply standard regularization methods in Hilbert spaces to this parameterization. Generalizations of this method to two-dimensional problems were presented in [9, 11, 12].

A realization of this method for the most general case of discontinuities as considered in [9] via a moving grid algorithm was developed in [14] (see also [13]). This algorithm yields good numerical results. However, in each iteration step of this method the whole grid is changed. To reduce the numerical effort the method of *adaptive grid regularization* was introduced in [15, 16]. Numerical results in these papers show that this method is an efficient and fast tool to identify discontinuities of solutions of ill-posed problems.

The idea of the method is to adjust not only the grid but also the regularizing norm after each iteration: Let $w \in L^1(\Omega)$ be a weight function satisfying

$$\int_\Omega w(\xi)\, d\xi = |\Omega|\,, \qquad w > 0 \text{ a.e.}\,, \qquad w^{-1} \in L^\infty(\Omega)\,. \tag{1.2}$$

Then the regularization is carried out in $\mathcal{X}_w$ defined as the Hilbert space $H^1(\Omega)$ or $H_0^1(\Omega)$ equipped with the (semi) norm

$$\|x\|_w := \int_\Omega |\nabla x(\xi)|^2 w^{-1}(\xi)\, d\xi\,.$$

If one uses Tikhonov regularization, then one lookes for a minimizer of

$$\|F(x) - y^\delta\|^2 + \alpha \|x - x_*\|_w^2$$

in $x_* + \mathcal{X}_h$, where $\mathcal{X}_h$ is a finite-dimensional subspace of $\mathcal{X}_w$ consisting of finite elements corresponding to a triangulation $\tau_h$ of $\Omega$. An appropriate choice are piecewise linear elements. $x_*$ usually plays the role of an initial guess. Since it can always be incorporated into $F$ or $y$, we assume w.l.o.g. in the following that $x_* = 0$.

Instead of Tikhonov regularization also the iteratively regularized Gauss-Newton method can be used to obtain good results especially for nonlinear problems (cf. [15]).

It turns out from regularization for curve representations that a somewhat optimal choice for $w$ would be obtained by minimizing

$$g_{\alpha,\beta}(x, w) := \|F(x) - y^\delta\|^2 + \alpha \int_\Omega (\beta^2 + |\nabla x(\xi)|^2)\, w^{-1}(\xi)\, d\xi \tag{1.3}$$

simultaneously with respect to $x$ and $w$ yielding that $w \sim \sqrt{\beta^2 + |\nabla x|^2}$. These considerations led us to the following algorithm:

**Algorithm 1.1. (Adaptive grid regularization)** Let $\alpha, \beta > 0$.

(i) Start with a uniform (rather coarse) grid in $\Omega$ yielding the triangulation $\tau_1$. Set $n := 1$, $x_0^\delta :\equiv 0$.

(ii) Compute a minimizer $x_n^\delta$ of

$$\begin{aligned} \tilde{g}_{\alpha,\beta}(x, w_n) &:= \|F(x) - y^\delta\|^2 + \alpha \int_\Omega |\nabla x(\xi)|^2\, w_n^{-1}(\xi)\, d\xi \\ w_n(\xi) &:= |\Omega| \sqrt{\beta^2 + |\nabla x_{n-1}^\delta(\xi)|^2} \Big/ \int_\Omega \sqrt{\beta^2 + |\nabla x_{n-1}^\delta(\xi)|^2}\, d\xi \end{aligned} \tag{1.4}$$

in the finite element space of piecewise linear functions $\mathcal{X}_n$ corresponding to the triangulation $\tau_n$.

(iii) If a stopping criterion is satisfied, the iteration is finished; otherwise:

(iv) Perform a local grid refinement

$$\tau_{n+1} := G(\tau_n, x_n^\delta)\,. \tag{1.5}$$

Set $n := n + 1$ and go to step (ii).

As grid refinement in (1.5) we have chosen the following procedure: triangles $T_i \in \tau_n$ are bisected whenever $\nabla x_n^\delta$ is large. This is done according to the following rules: a refinement is only performed if the size of the triangle is larger than a certain threshold, i.e.,

$$\operatorname{diam}(T_i) \geq h_{\min} \tag{1.6}$$

Under all these admissible triangles only those are refined where the corresponding weight $w_{n,i} := w_n|_{T_i}$, which is constant on each triangle $T_i$ (note that the finite elements are linear), is not smaller than the $k$-th largest weight and close enough to the largest weight, i.e.,

$$\begin{aligned} w_{n,i} &\geq \bar{w}_k := k\text{-th largest element among all } w_{n,j} \text{ with } \operatorname{diam}(T_i) \geq h_{\min}\,, && (1.7)\\ w_{n,i} &> \kappa * \max\{w_{n,j} \,:\, \operatorname{diam}(T_j) \geq h_{\min}\}\,. && (1.8) \end{aligned}$$

We want to mention that the Algorithm 1.1 (except for the local grid refinement) is similar to algorithms discussed in [5] ($F = Id$, $\nabla x$ is approximated by $L_\gamma x$), [6] ($F = Id$), and [3] ($F = Id$, infinite-dimensional setting, i.e., $\mathcal{X}_n = \mathcal{X}_{w_n}$, different penalty term).

## 2. Convergence Analysis

In this section we want to show that the sequence $\{x_n^\delta\}$ obtained by Algorithm 1.1 is convergent.

First of all, note that the subspaces $\mathcal{X}_n$ are increasing, i.e., $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}$. Due to the refinement condition (1.6) no refinement will occur anymore after some iteration step $\bar{n}$, i.e., $\mathcal{X}_n = \mathcal{X}_{\bar{n}}$ for all $n \geq \bar{n}$. It will turn out that under some conditions $x_n^\delta$ will converge towards a minimizer of

$$f_{\alpha,\beta}(x) := \|F(x) - y^\delta\|^2 + \alpha|\Omega|^{-1}\Big(\int_\Omega \sqrt{\beta^2 + |\nabla x(\xi)|^2}\,d\xi\Big)^2$$

in $\mathcal{X}_{\bar{n}}$ if $F$ is linear and at least towards a stationary point if $F$ is nonlinear.

If the threshold $h_{\min}$ in (1.6) is getting smaller and smaller, then obviously $\bar{n}$ is getting larger and larger. Therefore, we are also interested if such minimizers in $\mathcal{X}_{\bar{n}}$ converge to a minimizer in some infinite-dimensional space if $\bar{n}$ tends to infinity. To be able to show this, the penalty term above has to be replaced by a variational formulation that extends to non-smooth functions, i.e.,

$$f_{\alpha,\beta}(x) := \|F(x) - y^\delta\|^2 + \alpha|\Omega|^{-1}J_\beta(x)^2\,, \tag{2.1}$$

where

$$\begin{aligned} J_\beta(x) &:= \sup\{Q_\beta(x,v) \,:\, v \in [C_c^1(\Omega)]^d,\ \|v\|_\infty \leq 1\}\,, && (2.2)\\ Q_\beta(x,v) &:= \int_\Omega \Big(x(\xi)\operatorname{div} v(\xi) + \beta\sqrt{1 - |v(\xi)|^2}\Big)\,d\xi\,. && (2.3) \end{aligned}$$

Here, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ and $\|v\|_\infty := \sup|v(\xi)|$. It is easy to show that $J_\beta(x)$ coincides with $\int_\Omega \sqrt{\beta^2 + |\nabla x(\xi)|^2}\,d\xi$ if $x \in W^{1,1}(\Omega)$ (cf. [1]).

As usual, we say that a function $x \in L^1(\Omega)$ is of bounded variation if $J_0(x) < \infty$. It is well known that the space of all functions of bounded variation, $BV(\Omega)$, equipped with the norm

$$\|x\|_{BV} := \|x\|_{L^1} + |x|_{BV}\,, \qquad |x|_{BV} := J_0(x)\,, \tag{2.4}$$

is a Banach space. Obviously, $J_\beta(x) < \infty$ if and only if $J_0(x) < \infty$.

It was shown in [1] that, for any $\beta \geq 0$, $J_\beta(x)$ is convex and weakly lower semicontinuous with respect to the $L^p(\Omega)$ topology for $1 \leq p < \infty$.

To be able to guarantee the existence and stability of minimizers of (1.4) or (2.1) and for our convergence proofs we need some assumptions on the operator $F$:

**Assumption 2.1.** Let $\Omega$ be an open bounded convex subset of $\mathbb{R}^d$ $(d = 1, 2, 3)$ with Lipschitz boundary. The operator $F : \mathcal{D}(F) \subset L^p(\Omega) \to \mathcal{Y}$ is continuous, with $\mathcal{D}(F)$ convex and closed in $L^p(\Omega)$, for some $p < \bar{p}$ or continuous and weakly sequentially closed for $p = \bar{p}$ in case $d \geq 2$, where

$$\bar{p} := \begin{cases} \infty\,, & d = 1\,, \\ \frac{d}{d-1}\,, & d > 1\,. \end{cases}$$

(i) If $F$ is linear, then $\mathcal{D}(F) = L^p(\Omega)$.

(ii) If $F$ is nonlinear, it holds that $\mathcal{X}_n \cap \mathcal{D}(F) \neq \{\}$ and that $F$ is continuously Fréchet-differentiable from $\mathcal{X}_n \to L(\mathcal{X}_n, \mathcal{Y})$ for all $n \in \mathbb{N}$, where $\mathcal{X}_n$ is as in Algorithm 1.1. Moreover,

$$\|F(x) - F(\bar{x})\| \leq \rho(\|x - \bar{x}\|_{L^p})\,, \quad x, \bar{x} \in \mathcal{D}(F)\,, \tag{2.5}$$

for some continuous monotonically increasing function $\rho$. Furthermore, it holds for all constant functions $\kappa$ that $\kappa \in \mathcal{D}(F)$ and that

$$\|F(\kappa)\| \geq \bar{\gamma}|\kappa| \tag{2.6}$$

for some constant $\bar{\gamma} > 0$.

**Proposition 2.2.** *Let Assumption* 2.1 *hold. Then the functional* $\tilde{g}_{\alpha,\beta}$ *as defined in* (1.4) *has a minimizer* $x_n^\delta$ *in* $\mathcal{X}_n \cap \mathcal{D}(F)$ *and the functional* $f_{\alpha,\beta}$ *as defined in* (2.1) *has a minimizer* $x_{\alpha,\beta,n}^\delta$ *in* $\mathcal{X}_n \cap \mathcal{D}(F)$ *and a minimizer* $x_{\alpha,\beta}^\delta$ *in* $BV(\Omega) \cap \mathcal{D}(F)$*, respectively.*

*In case that* $F$ *is linear and injective the minimizer* $x_{\alpha,\beta}^\delta$ *is unique. If* $F$ *is linear, the minimizers* $x_n^\delta$ *and* $x_{\alpha,\beta,n}^\delta$ *are unique in case* $1 \notin \mathcal{N}(F)$ *and unique up to a constant otherwise.*

*Proof.* Let us consider functional $f_{\alpha,\beta}$ and let $\mathcal{Z}$ be either the space $\mathcal{X}_n \cap \mathcal{D}(F)$ or $BV(\Omega) \cap \mathcal{D}(F)$. Then there is a sequence $\{x_k\}$ in $\mathcal{Z}$ such that

$$\lim_{k\to\infty} f_{\alpha,\beta}(x_k) = \inf_{x\in\mathcal{Z}} f_{\alpha,\beta}(x) < \infty\,.$$

Therefore,

$$\|F(x_k)\| \leq \gamma \quad \text{and} \quad J_0(x_k) \leq \gamma\,, \qquad k \in \mathbb{N}\,,$$

where $\gamma > 0$ is a generic constant.

Let $\bar{x}_k := |\Omega|^{-1} \int_\Omega x_k(\xi)\, d\xi$. Then it follows from [2, Theorem 3.44] that

$$\|x_k - \bar{x}_k\|_{L^p} \le \gamma_p \|x_k - \bar{x}_k\|_{BV} \le \gamma J_0(x_k)\,.$$

Let us assume that condition (2.6) also holds if $F$ is linear, i.e., that $1 \notin \mathcal{N}(F)$. Then it follows that

$$\bar{\gamma}|\bar{x}_k| \le \|F(\bar{x}_k)\| \le \|F(x_k)\| + \|F(x_k) - F(\bar{x}_k)\|\,.$$

Together with (2.5) this implies that $|\bar{x}_k|$ is uniformly bounded. Now the estimate

$$\|x_k\|_{BV} \le |\Omega|\,|\bar{x}_k| + \gamma J_0(x_k)$$

yields that $\|x_k\|_{BV}$ is uniformly bounded.

If $F$ is linear and $1 \in \mathcal{N}(F)$, then obviously $f_{\alpha,\beta}(x_k + \kappa) = f_{\alpha,\beta}(x_k)$ for all constants $\kappa \in \mathbb{R}$. Therefore, we could have chosen $x_k$ from the very beginning such that $\bar{x}_k = 0$ which again yields the uniform boundedness of $\|x_k\|_{BV}$.

From an embedding theorem (cf. [2, Corollary 3.49]) we now obtain together with the uniform boundedness of $\|F(x_k)\|$ that $\{x_k\}$ has a subsequence, again denoted by $\{x_k\}$, such that

$$x_k \xrightarrow{L^p} x \quad \text{and} \quad F(x_k) \xrightarrow{\mathcal{Y}} F(x)$$

for some $x \in \mathcal{Z}$, where norm convergence has to be replaced by weak convergence if $p = \bar{p}$. Together with the weak lower semicontinuity of $f_{\alpha,\beta}$ this now yields that

$$\inf_{x\in\mathcal{Z}} f_{\alpha,\beta}(x) \le f_{\alpha,\beta}(x) \le \lim_{k\to\infty} f_{\alpha,\beta}(x_k) = \inf_{x\in\mathcal{Z}} f_{\alpha,\beta}(x)\,.$$

Thus, $x \in \mathcal{Z}$ is a minimizer.

The existence of a minimizer of $\tilde{g}_{\alpha,\beta}$ follows similarly. Note that, since $\mathcal{X}_n$ is finite-dimensional, all norms are equivalent on $\mathcal{X}_n$ and weak convergence already implies norm convergence.

It is obvious that the minimizers are unique if the functionals are strictly convex, which is the case if $F$ is linear and injective.

Let us now assume that $F$ is linear and that $1 \notin \mathcal{N}(F)$. Then strict convexity of $\tilde{g}_{\alpha,\beta}$ and $f_{\alpha,\beta}$ when considered over $\mathcal{X}_n$ follows from the fact that

$$\begin{aligned} J_\beta''(x)(h,h) &= \int_\Omega \frac{|\nabla h(\xi)|^2(\beta^2 + |\nabla x(\xi)|^2) - (\nabla x(\xi)^T \nabla h(\xi))^2}{(\beta^2 + |\nabla x(\xi)|^2)^{\frac{3}{2}}} \\ &\ge \beta^2 \int_\Omega \frac{|\nabla h(\xi)|^2}{(\beta^2 + |\nabla x(\xi)|^2)^{\frac{3}{2}}}\,. \end{aligned}$$

A similar estimate holds for the second derivative of the penalty term in $\tilde{g}_{\alpha,\beta}$. As already mentioned above the minimizers $x_n^\delta$ and $x_{\alpha,\beta,n}^\delta$ are unique up to a constant if $F$ is linear and $1 \in \mathcal{N}(F)$. ∎

We believe that the condition $F$ linear and $1 \notin \mathcal{N}(F)$ is also sufficient for the minimizer $x_{\alpha,\beta}^\delta \in BV(\Omega)$ to be unique, since we think that

$$J_\beta(\lambda x_1 + (1-\lambda)x_2) < \lambda J_\beta(x_1) + (1-\lambda) J_\beta(x_2)$$

also holds in the general case as long as $\beta > 0$ and $x_2 - x_1 \neq const$. However, so far we have not succeeded in proving it.

Results about stability follow similarly as in [7] and [1].

In the next theorem we will show our first convergence result in the finite-dimensional space $\mathcal{X}_{\bar{n}}$.

**Theorem 2.3.** *Let Assumption* 2.1 *hold and let* $x_n^\delta$ *be as in Algorithm* 1.1 *where the grid refinement is done according to* (1.6) – (1.8).

*Then* $\{x_n^\delta\}$ *has a convergent subsequence. The limit of every convergent subsequence is a stationary point of* $f_{\alpha,\beta}$*, defined by* (2.1)*, in case* $F$ *is nonlinear.*

*If* $F$ *is linear, the limit is even a minimizer* $x_{\alpha,\beta,\bar{n}}^\delta$ *of* $f_{\alpha,\beta}$ *in* $\mathcal{X}_{\bar{n}}$*. If this minimizer is unique, then* $x_n^\delta \to x_{\alpha,\beta,\bar{n}}^\delta$ *as* $n \to \infty$.

*Proof.* It is obvious from (1.3) and (1.4) that $x_n^\delta$ is not only a minimizer of $\tilde{g}_{\alpha,\beta}(x, w_n)$ but also of $g_{\alpha,\beta}(x, w_n)$. Since by the Cauchy-Schwarz inequality

$$\Big(\int_\Omega \sqrt{\beta^2 + |\nabla x(\xi)|^2}\, d\xi\Big)^2 \leq \int_\Omega \sqrt{\beta^2 + |\nabla x_{n-1}^\delta(\xi)|^2}\, d\xi \int_\Omega \frac{\beta^2 + |\nabla x(\xi)|^2}{\sqrt{\beta^2 + |\nabla x_{n-1}^\delta(\xi)|^2}}\, d\xi\,,$$

it follows with (1.4) and (2.1) that

$$f_{\alpha,\beta}(x_n^\delta) \leq g_{\alpha,\beta}(x_n^\delta, w_n) \leq g_{\alpha,\beta}(x_{n-1}^\delta, w_n) = f_{\alpha,\beta}(x_{n-1}^\delta)\,. \tag{2.7}$$

Thus, $f_{\alpha,\beta}(x_n^\delta)$ is monotonically decreasing and hence convergent. Moreover, $\|x_n^\delta\|_{BV}$ is uniformly bounded.

Let now $x_{n_k}^\delta$ be an arbitrary subsequence of $x_n^\delta$. Then there exists a further subsequence, again denoted by $x_{n_k}^\delta$, converging towards some $x \in \mathcal{X}_{\bar{n}}$ (in whatever norm, since all norms are equivalent). Note that then $\|w_n\|_{L^\infty} \leq \gamma$ for some $\gamma > 0$ and

$$\lim_{k\to\infty} w_{n_k+1}(\xi) = w(\xi) := |\Omega| \sqrt{\beta^2 + |\nabla x(\xi)|^2} \Big/ \int_\Omega \sqrt{\beta^2 + |\nabla x(\xi)|^2}\, d\xi\,. \tag{2.8}$$

It follows with (1.3), (2.7), the definition of $w_{n_k+1}$ (cf. (1.4)), and a Taylor expansion of $w^{-1}$ that (for some $\theta \in (0,1)$)

$$\begin{aligned}
f_{\alpha,\beta}(x_{n_k-1}^\delta) - f_{\alpha,\beta}(x_{n_k}^\delta) &\geq g_{\alpha,\beta}(x_{n_k}^\delta, w_{n_k}) - g_{\alpha,\beta}(x_{n_k}^\delta, w_{n_k+1}) \\
&= \alpha \int_\Omega (\beta^2 + |\nabla x_{n_k}^\delta(\xi)|^2)(w_{n_k}^{-1}(\xi) - w_{n_k+1}^{-1}(\xi))\, d\xi \\
&= \alpha \int_\Omega (\beta^2 + |\nabla x_{n_k}^\delta(\xi)|^2)\Big(\frac{w_{n_k+1}(\xi) - w_{n_k}(\xi))}{w_{n_k+1}(\xi)^2} \\
&\qquad\qquad + \frac{(w_{n_k}(\xi) - w_{n_k+1}(\xi))^2}{(w_{n_k+1}(\xi) + \theta(w_{n_k}(\xi) - w_{n_k+1}(\xi)))^3}\Big)\, d\xi \\
&= \alpha \int_\Omega (\beta^2 + |\nabla x_{n_k}^\delta(\xi)|^2) \frac{(w_{n_k}(\xi) - w_{n_k+1}(\xi))^2}{(w_{n_k+1}(\xi) + \theta(w_{n_k}(\xi) - w_{n_k+1}(\xi)))^3}\, d\xi \\
&\geq \alpha\beta^2\gamma^{-3} \|w_{n_k} - w_{n_k+1}\|_{L^2}^2\,.
\end{aligned}$$

Since the left hand side of the estimate above goes to 0, this together with (2.8) shows that also $w_{n_k}$ converges towards $w$. With the first order condition for a minimum of (1.4), i.e.,

$$\langle\, F(x_{n_k}^\delta) - y^\delta, F'(x_{n_k}^\delta)(z - x_{n_k}^\delta)\,\rangle + \alpha \int_\Omega \nabla x_{n_k}^\delta(\xi)^T (\nabla z(\xi) - \nabla x_{n_k}^\delta(\xi)) w_{n_k}^{-1}(\xi)\, d\xi \geq 0$$

for all $z \in \mathcal{D}(F) \cap \mathcal{X}_{\bar{n}}$ and $k \in \mathbb{N}$ sufficiently large (such that $\mathcal{X}_{n_k} = \mathcal{X}_{\bar{n}}$), we now obtain with $k \to \infty$ that

$$\langle F(x) - y^\delta, F'(x)(z-x) \rangle + \alpha \int_\Omega \nabla x(\xi)^T (\nabla z(\xi) - \nabla x(\xi)) w^{-1}(\xi)\, d\xi \geq 0$$

for all $z \in \mathcal{D}(F) \cap \mathcal{X}_{\bar{n}}$.

However, this is the first oder condition for a minimizer of (2.1) and hence $x$ is a stationary point. If $F$ is linear, the functional in (2.1) is convex and, therefore, the first order condition for a minimizer is also sufficient then, i.e., $x$ is a minimizer of $f_{\alpha,\beta}$.

If $x$ is unique, then obviously the whole sequence $x_n^\delta$ converges towards $x$ by a subsequence subsequence argument. ∎

Since $f_{\alpha,\beta}(x_n^\delta)$ is monotonically decreasing, the stationary point in the theorem above will never be a local maximum.

In the next theorem we will show that minimizers of (2.1) in $\mathcal{X}_n$ converge towards a minimizer of (2.1) in $BV(\Omega)$ if the spaces $\mathcal{X}_n$ approximate the space $BV(\Omega)$ in an appropriate way: let $P_n : BV(\Omega) \to \mathcal{X}_n$ be projection operators and let $\mathcal{P}_\beta \subset BV(\Omega)$ be defined as follows

$$\begin{aligned} \mathcal{P}_\beta := \{x \in BV(\Omega) \cap \mathcal{D}(F) : P_n x \overset{L^p}{\to} x\,,\ J_\beta(P_n x) \to J_\beta(x)\,, \\ P_n x \in \mathcal{D}(F) \text{ for } n \text{ sufficiently large}\} \end{aligned} \tag{2.9}$$

with $p$ as in Assumption 2.1.

**Theorem 2.4.** *Let Assumption* 2.1 *hold and assume that* $\mathcal{P}_\beta \neq \{\}$*, where* $\mathcal{P}_\beta$ *is defined as in* (2.9)*. Moreover, let* $x_{\alpha,\beta,n}^\delta$ *be a minimizer of* $f_{\alpha,\beta}$*, defined as in* (2.1)*, in* $\mathcal{X}_n$*.*

*Then* $\{x_{\alpha,\beta,n}^\delta\}$ *has a weakly* convergent subsequence in* $BV(\Omega)$*. The limit* $\tilde{x}$ *of every weakly* convergent subsequence satisfies the condition*

$$f_{\alpha,\beta}(\tilde{x}) \leq \liminf_{n\to\infty} f_{\alpha,\beta}(x_{\alpha,\beta,n}^\delta) \leq \limsup_{n\to\infty} f_{\alpha,\beta}(x_{\alpha,\beta,n}^\delta) \leq \inf_{x \in \mathcal{P}_\beta} f_{\alpha,\beta}(x)\,. \tag{2.10}$$

*If*

$$\mathcal{D}(F) \cap \bigcup_{n\in\mathbb{N}} \mathcal{X}_n \subset \mathcal{P}_\beta \tag{2.11}$$

*then it even holds that*

$$\lim_{n\to\infty} f_{\alpha,\beta}(x_{\alpha,\beta,n}^\delta) = \inf_{x\in\mathcal{P}_\beta} f_{\alpha,\beta}(\tilde{x})\,. \tag{2.12}$$

*If a minimizer* $x_{\alpha,\beta}^\delta$ *of* $f_{\alpha,\beta}$ *in* $BV(\Omega)$ *is an element of* $\mathcal{P}_\beta$*, then* $\tilde{x}$ *is also a minimizer of* $f_{\alpha,\beta}$ *in* $BV(\Omega)$ *and*

$$\lim_{n\to\infty} f_{\alpha,\beta}(x_{\alpha,\beta,n}^\delta) = \inf_{x\in BV\cap\mathcal{D}(F)} f_{\alpha,\beta}(\tilde{x})\,.$$

*If, in addition,* $x_{\alpha,\beta}^\delta$ *is unique, then* $\{x_{\alpha,\beta,n}^\delta\}$ *weakly* converges towards* $x_{\alpha,\beta}^\delta$ *and hence*

$$x_{\alpha,\beta,n}^\delta \overset{L^p}{\to} x_{\alpha,\beta}^\delta\,,\ \forall 1 \leq p < \bar{p}\,, \qquad \text{and} \qquad x_{\alpha,\beta,n}^\delta \overset{L^{\bar{p}}}{\rightharpoonup} x_{\alpha,\beta}^\delta\,,\ d \geq 2\,.$$

*Proof.* Let $x \in \mathcal{P}_\beta$. Then, due to Assumption 2.1, $f_{\alpha,\beta}(P_n x) \to f_{\alpha,\beta}(x)$. Since $f_{\alpha,\beta}(x_{\alpha,\beta,n}^\delta) \leq f_{\alpha,\beta}(P_n x)$ for $n$ sufficiently large, we now obtain that

$$\liminf_{n\to\infty} f_{\alpha,\beta}(x_{\alpha,\beta,n}^\delta) \leq \limsup_{n\to\infty} f_{\alpha,\beta}(x_{\alpha,\beta,n}^\delta) \leq \inf_{x\in\mathcal{P}_\beta} f_{\alpha,\beta}(x)\,.$$

As in the proof of Proposition 2.2 this yields that $\|F(x^\delta_{\alpha,\beta,n})\|$ and $\|x^\delta_{\alpha,\beta,n}\|_{BV}$ are uniformly bounded and, furthermore, that a weakly* convergent subsequence exists such that its limit $\tilde{x}$ satisfies (2.10).

If condition (2.11) holds, then $x^\delta_{\alpha,\beta,n} \in \mathcal{P}_\beta$ and hence

$$\inf_{x \in \mathcal{P}_\beta} f_{\alpha,\beta}(x) \le \liminf_{n\to\infty} f_{\alpha,\beta}(x^\delta_{\alpha,\beta,n}) ,$$

which together with (2.10) implies that (2.12) holds.

All other assertions are now obvious. Note that the convergence in $L^p$ follows from weak* convergence due to an embedding theorem already cited in the proof of Proposition 2.2. ∎

Of course, it would also be interesting to investigate what happens if $\alpha$, $\beta$, and/or $\delta$ converge to 0 or if $F$ is approximated by some $F_m$ as it is usual for nonlinear problems. Results can be derived similary as in [5, 7].

We would like to discuss now what it means that the set $\mathcal{P}_\beta$ is not empty or that a minimizer of $f_{\alpha,\beta}$ in $BV(\Omega)$ is an element of this set:

Note that condition (2.11) is not really restrictive: if for instance $\mathcal{X}_n \subset \mathcal{X}_{n+1}$ for all $n \in \mathbb{N}$, then obviously for all $x \in \bigcup_{n\in\mathbb{N}} \mathcal{X}_n$ it holds that $P_n x = x$ for $n$ sufficiently large. Hence, condition (2.11) is then satisfied if $P_n x \in \mathcal{D}(F)$ for $n$ sufficiently large.

We want to mention that condition

$$\liminf_{n\to\infty} f_{\alpha,\beta}(P_n x) \le f_{\alpha,\beta}(x) , \tag{2.13}$$

requested in [6, Theorem 2.2], only guarantees convergence of a subsequence and not that the whole sequence converges even if the minimizer $x^\delta_{\alpha,\beta}$ is unique. According to [6], condition (2.13) is satisfied for the $L^2$-orthogonal projectors $P_n$ if $x$ is smooth enough. This is correct but the solutions we are interested in have jumps and are therefore not smooth.

We will show below that in the one-dimensional case there is a projector $P_n$ such that $P_n x \overset{L^p}{\to} x$ and $J_\beta(P_n x) \to J_\beta(x)$ holds for all $x \in BV(\Omega)$. The proof of this result is essentially based on the following theorem:

**Theorem 2.5.** *Let the set $\mathcal{M}$ be defined as*

$$\mathcal{M} := \{\phi \in H^1[0,1] : \phi(0) = 0,\ \phi(1) = 1,\ \dot{\phi} \ge 0 \ a.e.\} . \tag{2.14}$$

*Then it holds that*

$$\exists(\phi, c) \in \mathcal{M} \times H^1[0,1] : x = c(\phi^{-1}) \ a.e. \quad \Longleftrightarrow \quad x \in BV[0,1] , \tag{2.15}$$

*where*

$$\phi^{-1}(s) := \inf\{t : \phi(t) = s\} . \tag{2.16}$$

*Moreover, if $x \in BV[0,1]$, then for every $\beta > 0$ the functions $\phi$ and $c$ in (2.15) may be even chosen such that*

$$\beta^2 \dot{\phi}^2 + \dot{c}^2 = J_\beta(x)^2 \ a.e. \qquad\qquad J_0(x) = \|\dot{c}\|_{L^1} \tag{2.17}$$

*holds. Thus, $\phi, c \in W^{1,\infty}[0,1]$.*

The proof of this theorem can be found in [10] (see also [9]). Since these references have not been published, a new proof based on the variational definition in (2.2) is given in the appendix for the convenience of the reader.

**Example 2.6.** We consider the space $BV[0,1]$ and the finite-dimensional subspaces $\mathcal{X}_n \subset H^1[0,1]$ of piecewise linear functions with triangulation

$$\tau_n := \{0 = s_0 < s_1 < \ldots < s_{n-1} < s_n = 1\}\,, \qquad h_n := \max_{1\le i\le n}(s_i - s_{i-1})\,,$$

and assume that $\lim\limits_{n\to\infty} h_n = 0$.

The projection operators $P_n : BV[0,1] \to \mathcal{X}_n$ are now defined as follows

$$\begin{array}{rclrcl} P_n x & := & \displaystyle\sum_{i=0}^{n} x_i v_i\,, & x_i & := & \displaystyle\lim_{\varepsilon\to 0} \tfrac{1}{2\varepsilon}\int_{s_i-\varepsilon}^{s_i+\varepsilon} x(\xi)\,d\xi\,, \quad 1 \le i < n\,, \\ x_0 & := & \displaystyle\lim_{\varepsilon\to 0^+} \tfrac{1}{\varepsilon}\int_0^{\varepsilon} x(\xi)\,d\xi\,, & x_n & := & \displaystyle\lim_{\varepsilon\to 0^+} \tfrac{1}{\varepsilon}\int_{1-\varepsilon}^{1} x(\xi)\,d\xi\,, \end{array} \tag{2.18}$$

where the functions $v_i$ are piecewise linear such that $v_i(s_j) = \delta_{ij}$.

The projectors $P_n$ are well defined as the following argument shows: it is well known (cf., e.g., [2, Theorem 3.28]) that for any $x \in BV[0,1]$ there is a unique function $\tilde{x}$ satisfying

$$x = \tilde{x} \text{ a.e.}\,, \qquad \tilde{x}(0) = \lim_{\xi\to 0}\tilde{x}(\xi)\,, \qquad \forall s \in (0,1] :\; \tilde{x}(s) = \lim_{\xi\to s^-}\tilde{x}(\xi)\,. \tag{2.19}$$

Thus, $\tilde{x}$ is of bounded variation with $J_0(x) = J_0(\tilde{x})$ and continuous from the left. Moreover, it holds that $\tilde{x}^+$ exists and that $\tilde{x}^+(s) = \tilde{x}(s)$ for all but at most countably many points. Here and below the superscript $^+$ for a function $f$ means that

$$\forall s \in [0,1) :\; f^+(s) := \lim_{\xi\to s^+} f(\xi)\,, \qquad f^+(1) := f(1)\,.$$

The values $x_i$ in (2.18) may be now expressed as

$$x_0 = \tilde{x}(0)\,, \qquad x_i = \tfrac{1}{2}(\tilde{x}(s_i) + \tilde{x}^+(s_i))\,, \quad 1 \le i < n\,, \qquad x_n = \tilde{x}(1)\,.$$

Now we use Theorem 2.5, i.e., let $\phi \in \mathcal{M}$ and $c \in H^1[0,1]$ be such that $x = c(\phi^{-1})$ a.e. and that (2.17) holds for some $\beta > 0$.

Due to the intermediate value theorem, there are values $\tilde{t}_i \in [\phi^{-1}(s_i), \phi^{-1}(s_i)^+]$ such that $x_i = c(\tilde{t}_i)$ (note that $\tilde{t}_0 = 0$, $\tilde{t}_n = 1$). Together with the transformation rule (see Remark A.9) we obatin that

$$\begin{aligned} \|(I-P_n)x\|_{L^p}^p &= \int_0^1 |c(\eta) - (P_n x)(\phi(\eta))|^p \dot{\phi}(\eta)\,d\eta \\ &= \sum_{i=1}^{n} \int_{\tilde{t}_{i-1}}^{\tilde{t}_i} \Big|\Big((c(\eta) - x_{i-1})(s_i - \phi(\eta)) \\ &\qquad\qquad + (c(\eta) - x_i)(\phi(\eta) - s_{i-1})\Big)(s_i - s_{i-1})^{-1}\Big|^p \dot{\phi}(\eta)\,d\eta \\ &\le \sum_{i=1}^{n} \Big(\int_{\tilde{t}_{i-1}}^{\tilde{t}_i} |\dot{c}(\eta)|\,d\eta\Big)^p (s_i - s_{i-1}) \\ &\le \Big(\int_0^1 |\dot{c}(\eta)|\,d\eta\Big)^p h_n\,, \end{aligned}$$

where we used that $\sum a_i^p \le (\sum a_i)^p$ (for $a_i \ge 0$, $p \ge 1$). Thus,

$$\|(I - P_n)x\|_{L^p} \le J_0(x)\, h_n^{\frac{1}{p}} \tag{2.20}$$

and hence $P_n x \xrightarrow{L^p} x$ for any $1 \le p < \infty$.

Using formula $\sqrt{\beta^2 a^2 + b^2}\sqrt{\beta^2 + e^2} \ge \beta^2 a + be$ (which can be easily shown), we obtain that

$$\begin{aligned}
&\sqrt{\beta^2\dot{\phi}^2(\eta) + \dot{c}^2(\eta)} - \sqrt{\beta^2 + ((P_n x)'(\phi(\eta)))^2}\dot{\phi}(\eta) \\
=\;& \frac{\sqrt{\beta^2\dot{\phi}^2(\eta) + \dot{c}^2(\eta)}\sqrt{\beta^2 + ((P_n x)'(\phi(\eta)))^2} - (\beta^2 + ((P_n x)'(\phi(\eta)))^2)\dot{\phi}(\eta)}{\sqrt{\beta^2 + ((P_n x)'(\phi(\eta)))^2}} \\
\ge\;& \frac{(P_n x)'(\phi(\eta))}{\sqrt{\beta^2 + ((P_n x)'(\phi(\eta)))^2}}(\dot{c}(\eta) - (P_n x)'(\phi(\eta))\dot{\phi}(\eta))
\end{aligned}$$

This together with (2.17) implies that

$$\begin{aligned}
J_\beta(x) - J_\beta(P_n x) &= \int_0^1 \sqrt{\beta^2\dot{\phi}^2(\eta) + \dot{c}^2(\eta)}\, d\eta - \int_0^1 \sqrt{\beta^2 + ((P_n x)'(\phi(\eta)))^2}\dot{\phi}(\eta)\, d\eta \\
&\ge \sum_{i=1}^n e_i \int_{\tilde{t}_{i-1}}^{\tilde{t}_i} (\dot{c}(\eta) - (P_n x)'(\phi(\eta))\dot{\phi}(\eta))\, d\eta \;=\; 0\,,
\end{aligned}$$

where $e_i := \frac{x_i - x_{i-1}}{\sqrt{\beta^2(s_i - s_{i-1})^2 + (x_i - x_{i-1})^2}}$. Note that $c(\tilde{t}_i) = x_i = P_n x(s_i) = P_n x(\phi(\tilde{t}_i))$ for all $0 \le i \le n$.

Thus, $J_\beta(P_n x) \le J_\beta(x)$ which together with the lower semicontinuity of $J_\beta$ and (2.20) implies that $J_\beta(P_n x) \to J_\beta(x)$.

This means that $\mathcal{P}_\beta = BV[0,1] \cap \mathcal{D}(F)$ if $P_n x \in \mathcal{D}(F)$ for $n$ sufficiently large which is always the case if $F$ is linear.

If $\mathcal{X}_n$ is chosen to be a subspace of $H_0^1[0,1]$, then the definition of the projection operators in (2.18) has to be adjusted, namely: $x_0 = 0 = x_n$.

Then it follows as above that $P_n x \xrightarrow{L^p} x$ for any $1 \le p < \infty$ and for all $x \in BV[0,1]$. However, $J_\beta(P_n x) \to J_\beta(x)$ only if $x \in BV[0,1]$ is such that

$$\tilde{x}(0) = 0 = \tilde{x}(1)\,.$$

To obtain convergence of $x^\delta_{\alpha,\beta,n}$ towards a minimizer $x^\delta_{\alpha,\beta}$ of $f_{\alpha,\beta}$ in $BV[0,1]$ according to Theorem 2.4 can, therefore, only be guaranteed if $x^\delta_{\alpha,\beta}$ satisfies the above condition. This means that only if $x^\delta_{\alpha,\beta}$ is 0 at the boundary it is advisable to choose $\mathcal{X}_n$ as subspace of $H_0^1[0,1]$.

Unfortunately, a similar result will not hold in higher dimensions. Even for piecewise constant functions, where jumps occur along lines of finite perimeter, it does not automatically hold that $J_\beta(P_n x) \to J_\beta(x)$. This is only the case if the triangulation approximates the line, where the jumps occur, good enough. In all other cases, one can only show that $J_\beta(P_n x)$ remains bounded.

It will be the topic of future research to investigate if other finite elements than piecewise linear ones are better suited to yield a result as in Example 2.6 also for higher dimensions. Moreover, we want to consider also other regularization methods than Tikhonov regularization in Algorithm 1.1.

Once more, we want to mention that numerical results confirming the theoretical ones can be found in [15, 16].

## Appendix A: Proof of Theorem 2.5

For the proof of Theorem 2.5 we need some preparatory work. In analogy to (2.3) and (2.2) we define for $\beta \geq 0$ and $0 \leq s_1 < s_2 \leq 1$:

$$Q_{\beta,s_1,s_2}(x,v) := \int_{s_1}^{s_2} \left( x(\xi)\dot{v}(\xi) + \beta\sqrt{1 - v^2(\xi)} \right) d\xi\,, \tag{A.1}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$, and

$$J_{\beta,s_1,s_2}(x) := \sup\{Q_{\beta,s_1,s_2}(x,v)\ :\ v \in C_c^1(0,1),\, \|v\|_\infty \leq 1\}\,. \tag{A.2}$$

Whenever used below, $\tilde{x}$ will be defined as in (2.19).

**Proposition A.7.** *Let $x \in BV[0,1]$ and $0 \leq s_1 < s_2 < s_3 \leq 1$. Then it holds that*

$$J_{\beta,s_1,s_3}(x) = J_{\beta,s_1,s_2}(x) + J_{\beta,s_2,s_3}(x) + |\tilde{x}^+(s_2) - \tilde{x}(s_2)|\,.$$

*Proof.* Let $v \in C_c^1(s_1,s_3)$ with $\|v\|_\infty \leq 1$ and such that the corresponding restrictions are still in $C_c^1(s_1,s_2)$ and $C_c^1(s_s,s_3)$, respectively, be arbitrary but fixed.

For $\varepsilon > 0$ sufficiently small such that $v|_{[s_2-2\varepsilon,s_2+2\varepsilon]} \equiv 0$ we define $v_\varepsilon$ as follows: $v_\varepsilon = v$ in $[s_1,s_3]\backslash[s_2-\varepsilon,s_2+\varepsilon]$ and $v_\varepsilon|_{[s_2-\varepsilon,s_2+\varepsilon]}$ is a piecewise linear function such that $v_\varepsilon(s_2-\varepsilon) = 0 = v_\varepsilon(s_2+\varepsilon)$ and $v_\varepsilon(s_2) = \mathrm{sgn}(\tilde{x}(s_2) - \tilde{x}^+(s_2))$. By locally smoothing the three corners, we can find a function $\tilde{v}_\varepsilon \in C_c^1(s_1,s_3)$ such that

$$\begin{aligned} Q_{\beta,s_1,s_3}(x,\tilde{v}_\varepsilon) \quad &\geq \quad Q_{\beta,s_1,s_2}(x,v) + Q_{\beta,s_2,s_3}(x,v) - \gamma_1\varepsilon \\ &\qquad + \Big(\tfrac{1}{\varepsilon}\int_{s_2-\varepsilon}^{s_2} x(\xi)\,d\xi - \tfrac{1}{\varepsilon}\int_{s_2}^{s_2+\varepsilon} x(\xi)\,d\xi\Big)\,\mathrm{sgn}(\tilde{x}(s_2) - \tilde{x}^+(s_2)) \end{aligned}$$

for some $\gamma_1 > 0$. Taking the limit $\varepsilon \to 0$, this yields together with (A.2) that

$$J_{\beta,s_1,s_3}(x) \geq J_{\beta,s_1,s_2}(x) + J_{\beta,s_2,s_3}(x) + |\tilde{x}^+(s_2) - \tilde{x}(s_2)|\,. \tag{A.3}$$

To finish the proof we will now show that the same estimate holds with $\leq$.

Let $v \in C_c^1(s_1,s_3)$ with $\|v\|_\infty \leq 1$ be arbitrary but fixed. For $\varepsilon > 0$ sufficiently small we define $v_\varepsilon$ as above, however, the piecewise linear part satisfies: $v_\varepsilon(s_2 \pm \varepsilon) = v(s_2 \pm \varepsilon)$ and $v_\varepsilon(s_2) = 0$. Again by locally smoothing the three corners, we can find a function $\tilde{v}_\varepsilon$ such that the corresponding restrictions are in $C_c^1(s_1,s_2)$ and $C_c^1(s_s,s_3)$, respectively. Moreover,

$$\begin{aligned} Q_{\beta,s_1,s_3}(x,v) \quad &\leq \quad Q_{\beta,s_1,s_2}(x,\tilde{v}_\varepsilon) + Q_{\beta,s_2,s_3}(x,\tilde{v}_\varepsilon) + \gamma_2\varepsilon \\ &\qquad + \Big(\tfrac{v(s_2-\varepsilon)}{\varepsilon}\int_{s_2-\varepsilon}^{s_2} x(\xi)\,d\xi - \tfrac{v(s_2+\varepsilon)}{\varepsilon}\int_{s_2}^{s_2+\varepsilon} x(\xi)\,d\xi\Big) \end{aligned}$$

for some $\gamma_2 > 0$ (depending on $v$). Taking the limit $\varepsilon \to 0$ we obtain together with (A.2) estimate (A.3) with $\geq$ replaced by $\leq$. ∎

**Corollary A.8.** *Let $x \in BV[0,1]$ and $0 \leq s_1 < s_2 < s_3 \leq 1$. Then it holds:*

(i) $J_{\beta,0,s}(x)$ *is continuous from the left as function in* $s$.

(ii) $J^+_{\beta,0,s}(x) = J_{\beta,0,s}(x) + |\tilde{x}^+(s) - \tilde{x}(s)|$ *for all* $0 < s < 1$ *and* $\lim_{s\to 0} J_{\beta,0,s}(x) = 0$.

(iii) $J_{\beta,0,s_2}(x) \geq J^+_{\beta,0,s_1}(x) + \beta(s_2 - s_1)$ *for all* $0 < s_1 < s_2 \leq 1$.

*Proof.* The assertion in (i) follows immediately from the definition (A.2) and the assertions in (ii) and (iii) follow from Proposition A.7 if we can show that

$$\lim_{h\to 0} J_{\beta,s,s+h}(x) = 0\,, \quad 0 \leq s < 1\,. \tag{A.4}$$

Let $0 \leq s < \bar{s} \leq 1$ and $\varepsilon > 0$ be arbitrary but fixed. Due to (A.2), there is a function $\tilde{v}_\varepsilon \in C^1_c[s, \bar{s}]$ with $\|\tilde{v}_\varepsilon\|_\infty \leq 1$ such that

$$J_{\beta,s,\bar{s}}(x) \leq Q_{\beta,s,\bar{s}}(x, \tilde{v}_\varepsilon) + \varepsilon\,.$$

Let now $\bar{h} > 0$ be such that $\tilde{v}_\varepsilon|_{[s,s+2\bar{h}]} \equiv 0$. Then it follows together with (A.2) and Proposition A.7 for all $0 < h \leq \bar{h}$ that

$$\begin{aligned} J_{\beta,s,s+h}(x) + J_{\beta,s+h,\bar{s}}(x) \;\leq\; J_{\beta,s,\bar{s}}(x) \;&\leq\; Q_{\beta,s,\bar{s}}(x, \tilde{v}_\varepsilon) + \varepsilon \\ &=\; Q_{\beta,s+h,\bar{s}}(x, \tilde{v}_\varepsilon) + \varepsilon \;\leq\; J_{\beta,s+h,\bar{s}}(x) + \varepsilon\,. \end{aligned}$$

Thus, $J_{\beta,s,s+h}(x) \leq \varepsilon$. Since $\varepsilon$ was arbitrary, this proves (A.4). ∎

**Remark A.9.** For the following considerations we need the transformation rule: Let $\phi \in \mathcal{M}$ (cf. (2.14)), $c, f \in C[0,1]$. Then it holds that

$$\int_0^1 f(\xi)\,d\xi = \int_0^1 f(\phi(\eta))\dot{\phi}(\eta)\,d\eta\,, \qquad \int_0^1 f(\xi)c(\phi^{-1}(\xi))\,d\xi = \int_0^1 f(\phi(\eta))c(\eta)\dot{\phi}(\eta)\,d\eta\,.$$

This is well known to hold for $\phi \in C^1[0,1]$. For $\phi \in \mathcal{M}$ the inverse is defined as in (2.16). From this definition we know that $\phi^{-1}$ is strictly monotonically increasing and continuous from the left. Moreover,

$$\phi(\phi^{-1}(s)) = s \quad \text{for all } s \in [0,1]\,, \tag{A.5}$$

$$\{\phi^{-1}(s)\} = \{t : \phi(t) = s\} \quad \text{for all but at most countably many } s \in [0,1]\,. \tag{A.6}$$

The transformation rules above now also hold for $\phi \in \mathcal{M}$, since one can find a sequence of functions $\phi_n \in C^1[0,1]$ such that $\phi_n(0) = 0$, $\phi_n(1) = 1$, $\dot{\phi}_n > 0$ and $\phi_n \to \phi$ as $n \to \infty$ and since the following lemma holds.

**Lemma A.10.** *Let* $\{\phi_n\}$ *be a sequence in* $\mathcal{M}$*. Then it holds that*

$$\phi_n \stackrel{H^1}{\rightharpoonup} \phi \quad \Longrightarrow \quad \phi_n^{-1}(s) \to \phi^{-1}(s) \ a.e.$$

*Proof.* Let $\{\phi_n\}$ be a sequence in $\mathcal{M}$ converging weakly in $H^1$ towards $\phi$. Since $\mathcal{M}$ is weakly closed in $H^1$, $\phi \in \mathcal{M}$. Let now $\{\phi_{n_k}\}$ be an arbitrary but fixed subsequence of $\{\phi_n\}$ and $s \in [0,1]$. Since $\phi_{n_k}^{-1}(s) \in [0,1]$, there exists a further subsequence denoted by $\{\phi_l\}$ and $t \in [0,1]$ with $\phi_l^{-1}(s) \to t$ as $l \to \infty$. Since, due to (A.5),

$$|s - \phi(t)| \leq |\int_t^{\phi_l^{-1}(s)} \dot{\phi}_l(\xi)\, d\xi| + |\phi_l(t) - \phi(t)| \leq \|\dot{\phi}_l\|_{L^2} |\phi_l^{-1}(s) - t|^{\frac{1}{2}} + \|\phi_l - \phi\|_{L^\infty}$$

and since $\|\dot{\phi}_l\|_{L^2}$ is uniformly bounded and $\|\phi_l - \phi\|_{L^\infty} \to 0$, we now obtain that $s = \phi(t)$. If $s$ satisfies (A.6), then $t = \phi^{-1}(s)$. Since $\{\phi_{n_k}\}$ was an arbitrary subsequence this implies that then $\phi_n^{-1}(s) \to \phi^{-1}(s)$. ∎

We are now in the position to prove Theorem 2.5:

*Proof of Theorem* 2.5. Let us first assume that the left hand side in (2.15) holds and that $v \in C_c^1[0,1]$ with $\|v\|_\infty \leq 1$. Then

$$Q_0(x,v) = \int_0^1 c(\eta)\dot{v}(\phi(\eta))\dot{\phi}(\eta)\, d\eta = -\int_0^1 \dot{c}(\eta) v(\phi(\eta))\, d\eta\,.$$

Thus, $J_0(x) \leq \|\dot{c}\|_{L^1}$ and hence $x \in BV[0,1]$.

Let now $\beta > 0$ and assume that $x \in BV[0,1]$. Then $J_\beta(x) < \infty$ and we may define $z$ as follows

$$z(0) := 0 \quad \text{and} \quad z(s) := \frac{J_{\beta,0,s}(x)}{J_\beta(x)}\,. \tag{A.7}$$

With Corollary A.8 we obtain that $z$ is continuous from the left, $z^+(0) = 0$, $z(1) = 1$, and

$$z(s_2) \geq z^+(s_1) + \tfrac{\beta}{J_\beta(x)}(s_2 - s_1)\,, \quad 0 \leq s_1 < s_2 \leq 1\,. \tag{A.8}$$

We now define $\phi$ as follows

$$\phi(t) := \inf\{s :\ t \leq z(s)\} \tag{A.9}$$

Then we immediately obtain together with (A.7) and (A.8) that $\phi$ is monotonically increasing, $\phi(0) = 0$, $\phi(1) = 1$, $0 < \phi(t) < 1$ for $0 < t < 1$, and

$$z(\phi(t)) = \underline{t} \leq t\,, \qquad z^+(\phi(t) = \overline{t} \geq t\,, \tag{A.10}$$

where

$$\underline{t} := \min\{\eta :\ \phi(\eta) = \phi(t)\} \qquad \overline{t} := \max\{\eta :\ \phi(\eta) = \phi(t)\}\,. \tag{A.11}$$

W.l.o.g. let $0 \leq t_1 < t_2 \leq 1$. Then $\phi(t_1) \leq \phi(t_2)$. If $\phi(t_1) < \phi(t_2)$, we obtain with (A.8) and (A.10) that

$$t_2 \geq z(\phi(t_2)) \geq z^+(\phi(t_1)) + \tfrac{\beta}{J_\beta(x)}(\phi(t_2) - \phi(t_1))$$

and furthermore that

$$0 \leq \phi(t_2) - \phi(t_1) \leq \tfrac{J_\beta(x)}{\beta}(t_2 - t_1)\,.$$

Therefore, $\phi$ is Lipschitz continuous and hence $\phi \in H^1[0,1]$ with $\dot{\phi} \geq 0$ a.e., i.e., $\phi \in \mathcal{M}$.

The function $c$ will be defined as follows

$$c(t) := \tilde{x}(\phi(t)) + (t - \underline{t}) J_\beta(x) \operatorname{sgn}(\tilde{x}^+(\phi(t)) - \tilde{x}(\phi(t))) \tag{A.12}$$

with $\underline{t}$ as in (A.11).

If $s_1 < s_2$, then it is easy to show (by choosing appropriate functions $\tilde{v}_\varepsilon$) that $J_{\beta,s_1,s_2}(x) \geq |\tilde{x}(s_2) - \tilde{x}^+(s_1)|$. This together with Proposition A.7, Corollary A.8, and (A.7) yields that

$$J_\beta(x)(z(s_2) - z^+(s_1)) \geq |\tilde{x}(s_2) - \tilde{x}^+(s_1)|\,, \quad J_\beta(x)(z^+(s_1) - z(s_1)) = |\tilde{x}^+(s_1) - \tilde{x}(s_1)|\,.$$

Together with (A.10) and (A.12) we obtain for $0 \leq t_1 < t_2 \leq 1$ that $c(\overline{t}_1) = \tilde{x}^+(\phi(t_1))$ and

$$\begin{aligned}
|c(t_2) - c(t_1)| &= |(t_2 - \underline{t}_2) J_\beta(x)\operatorname{sgn}(\tilde{x}^+(\phi(t_2)) - \tilde{x}(\phi(t_2))) + \tilde{x}(\phi(t_2)) - \tilde{x}^+(\phi(t_1)) \\
&\quad + (\overline{t}_1 - t_1) J_\beta(x)\operatorname{sgn}(\tilde{x}^+(\phi(t_1)) - \tilde{x}(\phi(t_1)))| \\
&\leq J_\beta(x)(t_2 - \underline{t}_2 + \underline{t}_2 - \overline{t}_1 + \overline{t}_1 - t_1) \;=\; J_\beta(x)(t_2 - t_1)
\end{aligned}$$

Therefore, $c$ is Lipschitz continuous and hence $c \in H^1[0,1]$.

Since $z$ is continuous from the left, $\phi^{-1}$ defined as in (2.16) satisfies that $\phi^{-1} = z$. Together with (A.5), (A.6), and (A.12) this implies that $c(\phi^{-1}) = \tilde{x}$ and hence, due to (2.19), $c(\phi^{-1}) = x$ a.e. This proves (2.15).

Let us now assume that $0 < s = \phi(t) \leq 1$. Then by definition (A.1), we get that

$$\begin{aligned}
Q_{\beta,0,s}(x,v) &= \int_0^s \Big( c(\phi^{-1}(\xi))\dot{v}(\xi) + \beta\sqrt{1 - v^2(\xi)} \Big)\, d\xi \\
&= \int_0^t \Big( c(\eta)\dot{v}(\phi(\eta))\dot{\phi}(\eta) + \beta\dot{\phi}(\eta)\sqrt{1 - v^2(\phi(\eta))} \Big)\, d\eta \\
&= \int_0^t \Big( -\dot{c}(\eta)v(\phi(\eta)) + \beta\dot{\phi}(\eta)\sqrt{1 - v^2(\phi(\eta))} \Big)\, d\eta
\end{aligned}$$

A density argument together with (A.2), (A.7), and (A.10) yields that

$$J_\beta(x)\,\underline{t} = J_{\beta,0,s}(x) = \sup_{\tilde{v}\in\mathcal{P}_1[0,t]} \int_0^t \Big( \dot{c}(\eta)\tilde{v}(\eta) + \beta\dot{\phi}(\eta)\sqrt{1 - \tilde{v}^2(\eta))} \Big)\, d\eta\,,$$

where $\mathcal{P}_1[0,t]$ is defined as follows: let $\mathcal{P}_2[0,t]$ be the set of piecewise constant functions $\tilde{v}$ defined on $[0,t]$ such that $\|\tilde{v}\|_\infty \leq 1$. Then $\mathcal{P}_1[0,t]$ is the subset of those functions $\tilde{v} \in \mathcal{P}_2[0,t]$ satisfying that $\tilde{v}|_{[\underline{t},t]} \equiv 0$ if $\underline{t} \neq t$ and that $\tilde{v}|_{[\underline{\eta},\overline{\eta}]} \equiv \mathit{const}$ for all $\eta \in (0,t)$ with $\underline{\eta} \neq \overline{\eta}$.

Since by definition $\phi$ is constant and $c$ is linear on all intervalls $[\underline{\eta},\overline{\eta}]$, it even follows that

$$J_\beta(x)\,\underline{t} = \sup_{\tilde{v}\in\mathcal{P}_2[0,t]} \int_0^t \Big( \dot{c}(\eta)\tilde{v}(\eta) + \beta\dot{\phi}(\eta)\sqrt{1 - \tilde{v}^2(\eta))} \Big)\, d\eta - J_\beta(x)(t - \underline{t})$$

and furthermore with a density argument that

$$J_\beta(x)\,t = \sup_{\tilde{v}\in\mathcal{P}_2[0,t]} \int_0^t \Big( \dot{c}(\eta)\tilde{v}(\eta) + \beta\dot{\phi}(\eta)\sqrt{1 - \tilde{v}^2(\eta))} \Big)\, d\eta = \int_0^t \sqrt{\beta^2\dot{\phi}^2(\eta) + \dot{c}^2(\eta)}\, d\eta\,.$$

This implies that $\beta^2\dot{\phi}^2 + \dot{c}^2 = J_\beta(x)^2$ a.e. Similarly it follows that $J_0(x) = \|\dot{c}\|_{L^1}$. ∎

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