Finding Telescopers with Minimal Depth for Indefinite Nested Sum and Product Expressions (Extended Version)

Carsten Schneider*
Research Institute for Symbolic Computation
J. Kepler University, A-4040 Linz, Austria
Carsten.Schneider@risc.uni-linz.ac.at

ABSTRACT

We provide algorithms that find, in case of existence, indefinite nested sum extensions in which a (creative) telescoping solution can be expressed with minimal nested depth.

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1. INTRODUCTION

Indefinite nested sums and products in rational terms can be represented in $\Pi\Sigma$-extensions [1]. More precisely, take a difference field $(\mathbb{G},\sigma)$, i.e., a field $\mathbb{G}$ together with a field automorphism $\sigma:\mathbb{G}\to\mathbb{G}$, and let $\mathbb{K}$ be its constant field, i.e., $\mathbb{K}=\text{const}_\sigma\mathbb{G}:=\{k\in\mathbb{G}\,|\,\sigma(k)=k\}$. Then a $\Pi\Sigma^*$-*extension* $(\mathbb{F},\sigma)$ of $(\mathbb{G},\sigma)$, a restricted version of $\Pi\Sigma$-extensions, is a difference field with $\text{const}_\sigma\mathbb{F}=\mathbb{K}$ of the following form: $\mathbb{F}=\mathbb{G}(t_1)\dots(t_e)$ is a rational function field and $\sigma:\mathbb{F}\to\mathbb{F}$ is extended from $\mathbb{G}$ to $\mathbb{F}$ by the recursively defined application $\sigma(t_i)=a_i\,t_i$ (product) or $\sigma(t_i)=t_i+a_i$ (sum) with $a_i\in\mathbb{G}(t_1)\dots(t_{i-1})$ for $1\leq i\leq e$. It is a Σ^*-*extension* (resp. Π-*ext.*), if for all t_i we have $\sigma(t_i)=t_i+a_i$ (resp. $\sigma(t_i)=a_i\,t_i$); $(\mathbb{F},\sigma)$ is a $\Pi\Sigma^*$-*field* over $\mathbb{K}$ if $\mathbb{G}=\mathbb{K}$.

Note that the nested depth of these sums/products gives a measure of the complexity of expressions. For instance, the sum of the left hand side of (2) has depth four, whereas the expression on the right hand side has only depth three.

In this article we try to reduce the depth of such indefinite sums by telescoping: First construct a $\Pi\Sigma^*$-field, say $(\mathbb{F},\sigma)$, in which the summand $f\in\mathbb{F}$ can be represented, and afterwards try to find a *telescoper* $g\in\mathbb{F}$ for $\sigma(g)-g=f$ where the depth of g is not larger than the depth of f. Given such a g, one gets, roughly speaking, $\sum_{k=0}^n f(k)=g(n+1)-g(0)$.

So far, methods have been developed in [1, 9] that assist in constructing a $\Pi\Sigma^*$-extension and in solving problem **A**, that covers besides telescoping also creative telescoping [11]. In particular, if $\mathbb{G}=\mathbb{K}$, i.e., $(\mathbb{F},\sigma)$ is a $\Pi\Sigma^*$-field, and if $\mathbb{K}$ is σ-computable[1], all these steps (constructing a $(\mathbb{F},\sigma)$ and solving **A**) can be done completely automatically; see [7].

A: *Given* a $\Pi\Sigma^*$-extension $(\mathbb{F},\sigma)$ of $(\mathbb{G},\sigma)$ and $\boldsymbol{f}=(f_1,\dots,f_n)\in\mathbb{F}^n$. *Find* all $g\in\mathbb{F}$ and $\boldsymbol{c}=(c_1,\dots,c_n)\in\mathbb{K}^n$ with

$$\sigma(g)-g=\boldsymbol{c}\boldsymbol{f},\quad\text{where } \boldsymbol{c}\boldsymbol{f}=\sum_{i=1}^n c_i\,f_i. \tag{1}$$

Within this approach it is crucial to find the appropriate $(\mathbb{F},\sigma)$ so that the depth can be reduced by telescoping. For instance, finding $(\mathbb{F},\sigma)$ is not so obvious, if one wants to reduce the depth of the left hand sides of

$$\sum_{k=1}^n\frac{1}{k^3}\sum_{j=1}^k\frac{H_j}{j^2}=H_n^{(3)}\sum_{j=1}^n\frac{H_j}{j^2}-\sum_{j=1}^n\frac{H_j(j^3H_j^{(3)}-1)}{j^5}, \tag{2}$$

$$\sum_{k=1}^n\left(\sum_{j=1}^k\frac{H_j^{(2)}}{j^3}\right)^2=-(H_n^{(2)2}+H_n^{(4)})\sum_{j=1}^n\frac{H_j^{(2)}}{j^3}+(n+1)\times$$
$$\times\left(\sum_{j=1}^n\frac{H_j^{(2)}}{j^3}\right)^2+\sum_{j=1}^n\frac{H_j^{(2)}\big((jH_j^{(2)})^2-H_j^{(2)}+j^2H_j^{(4)}\big)}{j^5};$$

we define $H_n:=\sum_{i=1}^n\frac{1}{i}$ and $H_n^{(r)}:=\sum_{i=1}^n\frac{1}{i^r}$, $r>1$. E.g., if one wants to find the right hand side of (2), one needs the sum extensions $H_n^{(3)}$ and $\sum_{j=1}^n H_j(j^3H_j^{(3)}-1)/j^5$ which do not occur on the left hand side; see Example 1.1. In short, using [1, 9] these two extensions must be adjoined manually.

Subsequently, we solve problem **B**, which dispenses the user completely from looking for the appropriate extension.

B: *Given* a $\Pi\Sigma^*$-extension $(\mathbb{F},\sigma)$ of $(\mathbb{G},\sigma)$, $\mathbb{K}:=\text{const}_\sigma\mathbb{F}$ and $\boldsymbol{f}\in\mathbb{F}^n$. *Decide* if there are $\boldsymbol{0}\neq\boldsymbol{c}\in\mathbb{K}^n$ and g in a $\Pi\Sigma^*$-extension $(\mathbb{F}(x_1)\dots(x_r),\sigma)$ of $(\mathbb{F},\sigma)$ such that (1) and[2] $\delta_\mathbb{G}(g)=\delta_\mathbb{G}(\boldsymbol{c}\boldsymbol{f})$ hold; if yes, compute such a solution.

Remark: In [8] we considered the special case $\sigma(x_i)-x_i\in\mathbb{F}$ which is too restricted to find the closed forms from above.

More precisely, we introduce *depth-optimal* $\Pi\Sigma^*$-extensions, a special class of $\Pi\Sigma^*$-extensions, in which we can represent constructively nested sums and products and in which we can solve **B**; see Theorems 6 and 8. It turns out that only Σ^*-extensions are needed to solve problem **B**.

The resulting algorithms are implemented in the summation package Sigma [7].

This technical report extends [10] by some additional proofs.

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[1] E.g., a rational function field $\mathbb{A}(x_1,\dots,x_r)$ over an algebraic number field $\mathbb{A}$ is σ-computable; for a definition see [5].

[2] $\delta_\mathbb{G}(f)$ denotes the maximal depth of the sums and products occurring in $f\in\mathbb{F}$ over the ground field $\mathbb{G}$; see Sec. 2.

2. PROPERTIES AND DEFINITIONS

First we work out that the problem to represent sums in Σ^*-extensions can be reduced to telescoping.

THEOREM 1. *[1] Let $(\mathbb{F}(t),\sigma)$ be a difference field.*
(1) *Then this is a Π-extension of $(\mathbb{F},\sigma)$ iff $\sigma(t)=a\,t$, $t\neq 0$, $a\in\mathbb{F}^*$ and there is no $n\neq 0$ and $g\in\mathbb{F}^*$ with $a^n=\frac{\sigma(g)}{g}$.*
(2) *Then this is a Σ^*-extension of $(\mathbb{F},\sigma)$ iff $\sigma(t)=t+a$, $t\notin\mathbb{F}$, $a\in\mathbb{F}$, and there is no $g\in\mathbb{F}$ with $\sigma(g)-g=a$.*

Namely, Theorem 1.2 shows that indefinite summation/telescoping and building up Σ^*-extensions are closely related. E.g., if one fails to find a $g\in\mathbb{F}$ with $\sigma(g)-g=f\in\mathbb{F}$, i.e., one cannot solve the telescoping problem in $\mathbb{F}$, one can adjoin the solution t with $\sigma(t)+t=f$ to $\mathbb{F}$ in form of the Σ^*-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$. Note that with similar techniques one can represent products in Π-extensions; see [5].

Summarizing, by solving **A**, nested sums can be represented in Σ^*-extensions. In Section 3 we will show that by refined telescoping we obtain also refined Σ^*-extensions. These new aspects can be illustrated as follows.

Example 1. **(1)** Given the left hand side of (2), say S_n, telescoping produces the $\Pi\Sigma^*$-field $(\mathbb{Q}(t_1)(t_2)(t_3)(t_4),\sigma)$ over $\mathbb{Q}$ with $\sigma(t_1)=t_1+1$, $\sigma(t_2)=t_2+\frac{1}{t_1+1}$ and $\sigma(t_{r+1})=t_{r+1}+\sigma(\frac{t_r}{t_1^r})$ for $r=2,3$. Namely, there is no $g\in\mathbb{Q}(t_1)$ with $\sigma(g)-g=\frac{1}{t_1+1}$ and no $g\in\mathbb{Q}(t_1)\dots(t_r)$ with $\sigma(g)-g=\sigma(\frac{t_r}{t_1^r})$ for $r=2,3$. Here t_4 represents S_n with depth 4. We can improve this situation by solving problem **B** for $\mathbb{F}:=\mathbb{Q}(t_1)(t_2)(t_3)$: We obtain the Σ^*-extension $(\mathbb{F}(x_1)(x_2),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(x_1)=x_1+\frac{1}{(t_1+1)^3}$, $\sigma(x_2)=x_2+\frac{(1+(t_1+1)t_2)x_1}{(t_1+1)^3}$ and $g:=x_1t_3-x_2$ such that $\sigma(g)-g=\sigma(\frac{t_3}{t_1^3})$; see Exp. 6.2. Then S_n is represented by g, which gives (2).
(2) Suppose we have represented S_n with $t_4\in\mathbb{Q}(t_1)\dots(t_4)$ as above, and suppose that we want to simplify S'_n given on the left hand side of (3). Then we adjoin the Σ^*-extension x_1 with $\sigma(x_1)=x_1+\frac{1}{(t_1+1)^3}$ in order to represent $H_j^{(3)}$, and look for a solution of $\sigma(g)-g=\frac{(1+(t_1+1)t_2)x_1}{(t_1+1)^3}$. In this case we compute $g=x_1t_3+t_4$ which is reflected by the identity

$$\sum_{j=1}^{n}\frac{H_j(j^3H_j^{(3)}-1))}{j^5}=H_n^{(3)}\sum_{j=1}^{n}\frac{H_j}{j^2}-\sum_{k=1}^{n}\frac{1}{k^3}\sum_{j=1}^{k}\frac{H_j}{j^2},\qquad(3)$$

i.e., we have increased the depth by telescoping!
This examples illustrates the advantages of $(\mathbb{F},\sigma)$ with $\mathbb{F}:=\mathbb{Q}(t_1)(t_2)(t_3)(x_1)(x_2)$: S_n can be represented by $x_1t_3-x_2$, and S'_n is given by x_2 with depth 3.

Finally, we introduce further definitions and properties.
• Let $(\mathbb{F},\sigma)$ be a difference field with $\mathbb{K}=\text{const}_\sigma\mathbb{F}$, $\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}^2$, $\boldsymbol{f}=(f_1,\dots,f_n)\in\mathbb{F}^n$ and $p\in\mathbb{F}$. We write $\sigma_{\boldsymbol{a}}p:=a_1\sigma(p)+a_2p$, $\sigma(\boldsymbol{f}):=(\sigma(f_1),\dots,\sigma(f_n))$ and $\boldsymbol{f}\,p:=(f_1\,p,\dots,f_n\,p)$. $\boldsymbol{a}$ is called *homogeneous over* $\mathbb{F}$ if $a_1\,a_2\neq 0$ and $\sigma_{\boldsymbol{a}}g=0$ for some $g\in\mathbb{F}^*$.
Let $\mathbb{V}$ be a subspace of $\mathbb{F}$ over $\mathbb{K}$ and suppose that $\boldsymbol{a}\neq\boldsymbol{0}$. Then we define the *solution space* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{V})$ as the subspace $\{(c_1,\dots,c_n,g)\in\mathbb{K}^n\times\mathbb{V}\,|\,\sigma_{\boldsymbol{a}}g=\sum_{i=1}^n c_if_i\}$ of the vector space $\mathbb{K}^n\times\mathbb{F}$ over $\mathbb{K}$. Note that the dimension is at most $n+1$; see [1]. *Summarizing, problem* **A** *is solved if one finds a basis of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$.
• Let $(\mathbb{G}(t_1)\dots(t_e),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G},\sigma)$ with $\sigma(t_i)=a_i\,t_i$ or $\sigma(t_i)=t_i+a_i$. Then the *depth-function over* $\mathbb{G}$, $\delta_\mathbb{G}:\mathbb{G}(t_1)\dots(t_e)\to\mathbb{N}_0$, is defined as follows. For any $g\in\mathbb{G}$ set $\delta(g)=0$. If $\delta_\mathbb{G}$ is defined for $(\mathbb{G}(t_1)\dots(t_{i-1}),\sigma)$ with $i>1$, we define $\delta(t_i)=\delta_\mathbb{G}(a_i)+1$, and for $g\in\mathbb{G}(t_1)\dots(t_i)$ we define[3] $\delta(g)=\max(\{\delta_\mathbb{G}(t_i)\,|\,t_i \text{ occurs in } g\}\cup\{0\})$. We define $\delta_\mathbb{G}(\boldsymbol{f})=\max_i\delta_\mathbb{G}(f_i)$ for $\boldsymbol{f}=(f_1,\dots,f_n)\in\mathbb{F}^n$. The *depth of* $(\mathbb{F},\sigma)$ *over* $\mathbb{G}$, $\delta_\mathbb{G}(\mathbb{F})$, is defined by $\delta_\mathbb{G}((0,t_1,\dots,t_e))$.
Convention: Throughout this article the depth is defined over $(\mathbb{G},\sigma)$; we set $\delta:=\delta_\mathbb{G}$. We might use the depth-function without mentioning $\mathbb{G}$. Then we assume that the corresponding difference fields are $\Pi\Sigma^*$-extensions of $(\mathbb{G},\sigma)$. In all our examples we will assume that $\mathbb{G}=\mathbb{Q}$.
• Let $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=\alpha_i\,t_i+\beta_i$. This extension is called ordered if $\delta(t_i)\leq\delta(t_{i+1})$. The extension has *maximal depth* $\mathfrak{d}$ if $\delta(t_i)\leq\mathfrak{d}$.
If there is a permutation $\tau:\{1,\dots,e\}\to\{1,\dots,e\}$ with $\alpha_{\tau(i)},\beta_{\tau(i)}\in\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(i-1)})$ for all $1\leq i\leq e$, then the generators of the $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ can be reordered without changing the $\Pi\Sigma^*$-nature of the extension. In short, we say that $(\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(e)}),\sigma)$ can be *reordered to* $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ if there exists such a τ. On the rational function field level we identify two such fields.
• Let $(\mathbb{F},\sigma)$ and $(\mathbb{F}',\sigma')$ be difference fields. Then a *σ-monomorphism/σ-isomorphism* is a field monomorphism/isomorphism $\tau:\mathbb{F}\to\mathbb{F}'$ with $\sigma'(\tau(a))=\tau(\sigma(a))$ for all $a\in\mathbb{F}$. Suppose that $(\mathbb{F},\sigma)$ is a $\Pi\Sigma^*$-ext. of $(\mathbb{H},\sigma)$.
An *$\mathbb{H}$-monomorphism/$\mathbb{H}$-isomorphism* $\tau:\mathbb{F}\to\mathbb{F}'$ is a σ-monomorphism/σ-isomorphism with $\tau(a)=a$ for all $a\in\mathbb{H}$.

LEMMA 1. *Let $(\mathbb{F}(x),\sigma)$ with $\sigma(x)=\alpha x+\beta$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$.* **(1)** *Let $a,f\in\mathbb{F}$ and suppose there is a solution $g\in\mathbb{F}(x)$ with $\sigma(g)-ag=f$, but no solution in $\mathbb{F}$. If x is a Π-extension then $f=0$ and $a=\frac{\sigma(h)}{h}\alpha^m$ for some $h\in\mathbb{F}$, $m\neq 0$; if x is a Σ^*-extension then $f\neq 0$ and $a=1$.*
(2) *Let $(\mathbb{F}(x)(t),\sigma)$ be a Π-extension of $(\mathbb{F}(x),\sigma)$ with $\alpha':=\sigma(t)/t\in\mathbb{F}$. Let $(a_1,a_2)\in\mathbb{F}$ be homogeneous over $\mathbb{F}$, $\boldsymbol{a'}:=(a_1\,\alpha'^i,a_2)$ with $i\neq 0$ and $\boldsymbol{f}\in\mathbb{F}^n$. Then $\boldsymbol{a'}$ is inhomogeneous over $\mathbb{F}(x)$ and $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{F})=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{F}(x))$.*

PROOF. **(1)** is equivalent to [2, Lemmas 4.1,4.2].
(2) Consider the extensions x and t as above, and let (a_1,a_2) be homogeneous, i.e., we can take a $g\in\mathbb{F}^*$ with $-a_2/a_1=\sigma(g)/g$. Suppose that $\boldsymbol{a'}$ is homogeneous over $\mathbb{F}(x)$, i.e., there is an $h\in\mathbb{F}(x)^*$ with $-a_2/(a_1\alpha'^i)=\sigma(h)/h$. Then $\alpha'^i=\sigma(g/h)/(g/h)$, and hence t is not a Π-ext. by Thm. 1.1; a contradiction. Now suppose $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{F})\subsetneq\mathrm{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{F}(x))$. If x is a Π-extension then by Lemma 1.1 there is a $p\in\mathbb{F}^*$ with $-a_2/(a_1\alpha'^i)\alpha^m=\frac{\sigma(p)}{p}$ for some $m\neq 0$. This implies that $\alpha^m=\sigma(h)/h$ with $h=t^ip/g$. By Thm. 1.1 $(\mathbb{F}(t)(x),\sigma)$ is not a Π-extension of $(\mathbb{F}(t),\sigma)$, a contradiction by reordering of $(\mathbb{F}(x)(t),\sigma)$. Otherwise, if x is a Σ^*-ext. then by Lemma 1.1, $-a_2/(a_1\alpha'^i)=1$, i.e., $\sigma(g)/g=\alpha'^i$. By Thm. 1.1 $(\mathbb{F}(t),\sigma)$ is not a Π-ext. of $(\mathbb{F},\sigma)$, a contradiction. Summarizing, $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{F})=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{F}(x))$. $\square$

PROPOSITION 1. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\mathbb{K}:=\text{const}_\sigma\mathbb{F}$. Then the following holds:*
(1) *Let $\boldsymbol{a}\in\mathbb{F}^2$ be homogeneous over $\mathbb{F}$ and $f\in\mathbb{F}$. If there is a $g\in\mathbb{E}\setminus\mathbb{F}$ with $\sigma_{\boldsymbol{a}}g=f$ then there is not such a g in $\mathbb{F}$.*
(2) *Let $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ with $\sigma(t_i)-t_i\in\mathbb{F}$ or $\frac{\sigma(t_i)}{t_i}\in\mathbb{F}$. If $\sigma(g)-g=f$ for $g\in\mathbb{E}$ then $g=\sum_{i=1}^e c_i\,t_i+w$ where $c_i\in\mathbb{K}$ and $w\in\mathbb{F}$; moreover, $c_i=0$, if $\frac{\sigma(t_i)}{t_i}\in\mathbb{F}$.*
(3) *Let $(\mathbb{F}(t),\sigma)$ and $(\mathbb{F}(t'),\sigma)$ be Σ^*-extensions of $(\mathbb{F},\sigma)$*

[3] g is given by $g=g_1/g_2$ with $g_1,g_2\in\mathbb{G}[t_1,\dots,t_i]$ coprime.

with $g \in \mathbb{F}(t') \setminus \mathbb{F}$ s.t. $\sigma(g) - g = \sigma(t) - t$. Then there is an $\mathbb{F}$-isomorphism $\tau : \mathbb{F}(t) \to \mathbb{F}(t')$ with $\tau(t) = g$.
(4) *Let $\tau : \mathbb{F} \to \mathbb{F}'$ be a σ-isomorphism for $(\mathbb{F}, \sigma), (\mathbb{F}', \sigma)$. Then there is a $\Pi\Sigma^*$-extension $(\mathbb{E}', \sigma)$ of $(\mathbb{F}', \sigma)$ with a σ-isomorphism $\tau' : \mathbb{E} \to \mathbb{E}'$ where $\tau'(a) = \tau(a)$ for all $a \in \mathbb{F}$.*

Proof. **(1)** Assume there are such $g' \in \mathbb{F}$ and $g \in \mathbb{E} \setminus \mathbb{F}$. Then $\sigma_{\boldsymbol{a}}(g - g') = 0$. By assumption $\sigma_{\boldsymbol{a}} h = 0$ for some $h \in \mathbb{F}^*$. Hence $\sigma(\frac{g-g'}{h}) = \frac{g-g'}{h}$, and thus $\text{const}_\sigma \mathbb{E} \neq \text{const}_\sigma \mathbb{F}$, a contradiction that $(\mathbb{E}, \sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$. **(2)** The second statement follows by Karr' Fundamental Theorem [2, Result, page 314]; see also [3, Thm 4.2.1]. **(3)** Let $\sigma(g) - g = \sigma(t) - t =: \beta \in \mathbb{F}$. By Prop. 1.2 there are a $c \in \text{const}_\sigma \mathbb{F}$ and a $w \in \mathbb{F}$ such that $g = c\,t' + w$. Since t' is transcendental over $\mathbb{F}$, also g is transcendental over $\mathbb{F}$ and therefore $\tau : \mathbb{F}(t) \to \mathbb{F}(g)$ canonically defined by $\tau(t) = g$ is a field isomorphism. We have $\tau(\sigma(t)) = \tau(t + \beta) = g + \beta = \sigma(g) = \sigma(\tau(t))$ and thus τ is an $\mathbb{F}$-isomorphism. Since $\mathbb{F}(g) = \mathbb{F}(s)$, the third part is proven. **(4)** Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\,t + \beta$. Since τ is a σ-isomorphism, there is a $\Pi\Sigma^*$-ext. $(\mathbb{F}'(t'), \sigma)$ of $(\mathbb{F}', \sigma')$ with $\sigma(t') = \tau(\alpha)t' + \tau(\beta)$ by Thm. 1. Take the field isomorphism $\tau' : \mathbb{F}(t) \to \mathbb{F}'(t')$ with $\tau'(t) := t'$ and $\tau'(a) := \tau(a)$ for all $a \in \mathbb{F}$. Since $\sigma(\tau(t)) = \tau(\sigma(t))$, τ' is a σ-isomorphism. Iterative application proves the last statement. $\square$

3. DEPTH-OPTIMAL $\Pi\Sigma^*$-EXTENSIONS

In this section we introduce depth-optimal $\Pi\Sigma^*$-extensions and motivate its relevance to symbolic summation. Afterwards we show how the problem to represent sums in δ-optimal extensions and how problem **B** can be reduced to problem **C** given below. To this end, we develop algorithms that solve **C** in Section 5.
A Σ^*-extension $(\mathbb{F}(s), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(s) = s + f$ is called *depth-optimal*, in short *δ-optimal*, if there is no $\Pi\Sigma^*$-extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}, \sigma)$ with maximal depth $\delta(f)$ such that $\sigma(g) - g = f$ holds for some $g \in \mathbb{H} \setminus \mathbb{F}$. A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1) \ldots (t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ is called *δ-optimal* if all the Σ^*-extensions are δ-optimal.
First we give some examples.

Lemma 2. *A $\Pi\Sigma^*$-extension $(\mathbb{G}(t_1) \ldots (t_e), \sigma)$ of $(\mathbb{G}, \sigma)$ with $\delta(t_i) \leq 2$, $\sigma(t_1) = t_1 + 1$ and $\text{const}_\sigma \mathbb{G} = \mathbb{G}$ is δ-optimal.*

Proof. If $\delta(t_k) = 1$, t_k is δ-optimal. Otherwise, if $\delta(t_k) = 2$ and t_k is not δ-optimal, then $\beta := \sigma(t_k) - t_k \in \mathbb{F}$ for $\mathbb{F} := \mathbb{G}(t_1) \ldots (t_{k-1})$ and there is a $\Pi\Sigma^*$-ext. $(\mathbb{F}(x_1) \ldots (x_r), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\delta(x_i) = 1$ and $g \in \mathbb{F}(x_1) \ldots (x_r) \setminus \mathbb{F}$ s.t. $\sigma(g) - g = \beta$. By Prop. 1.2, $q_j := \sigma(x_j) - x_j \in \mathbb{G}$ for some x_j. Then $\sigma(q_j t_1) - q_j t_1 = q_j$, i.e., x_j is not a Σ^*-extension by Thm. 1; a contradiction. $\square$

Example 2. Consider the $\Pi\Sigma^*$-field $(\mathbb{Q}(t_1)(t_2)(t_3)(t_4), \sigma)$ from Exp. 1.1. t_1, t_2 are δ-optimal extensions by Lemma 2. Moreover, t_3 is δ-optimal by Exp. 4. t_4 is not δ-optimal since we find the extension $(\mathbb{Q}(t_1)(t_2)(t_3)(x_1)(x_2), \sigma)$ and $g := x_1 t_3 - x_2$ s.t. $\sigma(g) - g = \sigma(t_4) - t_4$. Later we will see that the reordered extension $(\mathbb{Q}(t_1)(t_2)(x_1)(t_3)(x_2), \sigma)$ is δ-optimal; see Exp. 5 for x_1 and Exp. 6.2 for x_2.

Next, we work out some important properties.
• In Example 1.2 we have illustrated that in a $\Pi\Sigma^*$-extension $(\mathbb{F}, \sigma)$ of $(\mathbb{G}, \sigma)$ we might arrive at a solution $g \in \mathbb{F}$ of $\sigma(g) - g = f$ with $f \in \mathbb{F}$ where $\delta(g) > \delta(f) + 1$. This bad situation cannot happen in δ-optimal extensions; see Theorem 2.

Lemma 3. *Let $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{F}(t_1) \ldots (t_e)$ be a δ-optimal ordered $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ where $\mathfrak{d} := \delta(\mathbb{F})$, $\delta(t_i) > \mathfrak{d}$.* **(1)** *If $\sigma(g) - g = f$ for $g \in \mathbb{E}$ and $f \in \mathbb{F}$ with $\delta(f) < \mathfrak{d}$, then $\delta(g) \leq \mathfrak{d}$.* **(2)** *If $(\mathbb{F}(x_1) \ldots (x_r), \sigma)$ is a Σ^*-extension of $(\mathbb{F}, \sigma)$ with $\beta_i := \sigma(x_i) - x_i$ and $\delta(x_i) \leq \mathfrak{d}$, then there is the Σ^*-extension $(\mathbb{E}(x_1) \ldots (x_r), \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(x_i) = x_i + \beta_i$.*

Proof. **(1)** Suppose we have $\sigma(g) - g = f$ with $g \in \mathbb{E}$ and $m := \delta(g) > \mathfrak{d}$. By Prop. 1.2, $g = \sum_{i=1}^k c_i\, t_i + h$ where t_k is a Σ^*-ext. with $\delta(t_k) = m$ and $c_k \neq 0$. Set $\mathbb{H} := \mathbb{F}(t_1) \ldots (t_{k-1})$. By Prop. 1.1 there is no $g' \in \mathbb{H}^*$ with $\sigma(g') - g' = f$. Therefore by Thm. 1 one can construct a Σ^*-ext. $(\mathbb{H}(s), \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(s) = s + f$ where $\delta(s) = \delta(f) + 1 \leq \mathfrak{d} < m$. Note that $\sigma(g') - (g') = \sigma(t_k) - t_k$ with $g' = (s - \sum_{i=1}^{k-1} c_i t_i)/c_k \in \mathbb{H}(s)$. Hence t_k is not δ-optimal, a contradiction. Therefore $\delta(g) \leq \mathfrak{d}$. **(2)** For $r = 0$ we are done. Otherwise, let $i \geq 1$ be minimal s.t. $(\mathbb{E}(x_1) \ldots (x_i), \sigma)$ is not a Σ^*-ext. of $(\mathbb{E}, \sigma)$. Then there is a $g \in \mathbb{E}(x_1) \ldots (x_{i-1})$ with $\sigma(g) - g = \beta_i$. By Lemma 3.1 it follows that $\delta(g) \leq \mathfrak{d}$, i.e., $g \in \mathbb{F}(x_1) \ldots (x_{i-1})$, a contradiction to Thm. 1.2. $\square$

Theorem 2. *Suppose that $(\mathbb{F}, \sigma)$ is a δ-optimal ordered $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$ and $f \in \mathbb{F}^*$. If $\sigma(g) - g = f$ for $g \in \mathbb{F}^*$ then $\delta(f) \leq \delta(g) \leq \delta(f) + 1$.*

Proof. Since $\delta(\sigma(g) - g) \leq \delta(g)$, $\delta(f) \leq \delta(g)$. If $\delta(\mathbb{F}) = \delta(f)$, $\delta(g) = \delta(f)$. Otherwise, take the δ-optimal ordered $\Pi\Sigma^*$-ext. $(\mathbb{F}, \sigma)$ of $(\mathbb{H}, \sigma)$ with $\mathbb{F} = \mathbb{H}(t_1) \ldots (t_e)$, $\delta(\mathbb{H}) = \delta(f) + 1$, $\delta(t_i) > \delta(f) + 1$. By Lemma 3.1, $\delta(g) \leq \delta(f) + 1$. $\square$

Remark: In order to find all solutions of (1) in a δ-optimal ordered $\Pi\Sigma^*$-extension, one only has to consider those extensions whose depth is smaller or equal to $\delta(\boldsymbol{f}) + 1$.

• We show a reordering property; the general case that reordering gives again a δ-optimal extension is skipped here.

Lemma 4. *Let $(\mathbb{F}(t_1) \ldots (t_e)(x), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ where $(\mathbb{F}(t_1) \ldots (t_e), \sigma)$ is a δ-optimal extension of $(\mathbb{F}, \sigma)$ and $\delta(x) < \delta(t_i)$. By reordering $(\mathbb{F}(x)(t_1) \ldots (t_e), \sigma)$ is a δ-optimal $\Pi\Sigma^*$-extension of $(\mathbb{F}(x), \sigma)$.*

Proof. If $e = 0$ nothing has to be shown. Suppose that the lemma holds for $e \geq 0$. Consider $(\mathbb{F}(t_1) \ldots (t_e)(x), \sigma)$ as claimed above with $e > 0$. Then by the induction assumption $(\mathbb{F}(t_1)(x)(t_2) \ldots (t_e), \sigma)$ is a δ-optimal $\Pi\Sigma^*$-extension of $(\mathbb{F}(t_1)(x), \sigma)$. Note that $(\mathbb{F}(x)(t_1), \sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$. If t_1 is a Π-extension, we are done. Otherwise, suppose that t_1 is a Σ^*-extension which is not δ-optimal. Then there is a $\Pi\Sigma^*$-extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}(x), \sigma)$ with maximal depth $\delta(t_1) - 1$ and $g \in \mathbb{H}$ with $\sigma(g) - g = \sigma(t_1) - t_1$. Since $\delta(x) < \delta(t_1)$, $(\mathbb{H}, \sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with maximal depth $\delta(t_1) - 1$. Consequently, $(\mathbb{F}(t_1), \sigma)$ is not a δ-optimal Σ^*-extension of $(\mathbb{F}, \sigma)$, a contradiction. $\square$

• Now we can show that a δ-optimal Σ^*-extension $(\mathbb{S}, \sigma)$ of $(\mathbb{F}, \sigma)$ is "depth-optimal": Given a $\Pi\Sigma^*$-extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}, \sigma)$, one can construct a $\Pi\Sigma^*$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{S}, \sigma)$ in which the elements of $\mathbb{H}$ can be embedded by an $\mathbb{F}$-monomorphism $\tau : \mathbb{H} \to \mathbb{E}$ without increasing the depth.
Remark. $(\mathbb{E}, \sigma)$ and τ can be computed, if one can solve **A**.

Theorem 3. *Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$; let $(\mathbb{S}, \sigma)$ be a Σ^*-extension of $(\mathbb{F}, \sigma)$ which gives a δ-optimal ordered extension of $(\mathbb{G}, \sigma)$ by reordering. Then for any $\Pi\Sigma^*$-ext. $(\mathbb{H}, \sigma)$ of $(\mathbb{F}, \sigma)$ with maximal depth d there is a $\Pi\Sigma^*$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{S}, \sigma)$ with maximal depth d and an $\mathbb{F}$-monomorphism $\tau : \mathbb{H} \to \mathbb{E}$ where $\delta(\tau(a)) \leq \delta(a)$ for $a \in \mathbb{H}$.*

Proof. Let $(\mathbb{D},\sigma)$ be the δ-optimal ordered $\Pi\Sigma^*$-ext. of $(\mathbb{F},\sigma)$ that we get by reordering the Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{F},\sigma)$. Moreover, let $(\mathbb{H},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with maximal depth d, i.e., $\mathbb{H} := \mathbb{F}(t_1)\dots(t_e)$ with $d_i := \delta(t_i) \leq d$. Suppose that $\delta(t_i) \leq \delta(t_{i+1})$, otherwise we can reorder it without loosing any generality. We will show that there is a $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{D},\sigma)$ with maximal depth d and an $\mathbb{F}$-monomorphism $\tau : \mathbb{H} \to \mathbb{E}$ with $\delta(\tau(a)) \leq \delta(a)$ for all $a \in \mathbb{H}$. Then reordering of $(\mathbb{D},\sigma)$ proves the corresponding result for the extension $(\mathbb{S},\sigma)$ of $(\mathbb{F},\sigma)$. Besides this we will show that there is a Σ^*-extension $(\mathbb{A},\sigma)$ of $(\mathbb{H},\sigma)$ and a σ-isomorphism $\rho : \mathbb{E} \to \mathbb{A}$ such that $\rho(\tau(a)) = a$ for all $a \in \mathbb{F}(t_1)\dots(t_e)$; this property is needed to handle the Π-extension case in the proof step (II).

Induction base: If $e = 0$, i.e., $\mathbb{H} = \mathbb{F}$, the statement is proven by taking $(\mathbb{E},\sigma) := (\mathbb{D},\sigma)$ with the $\mathbb{F}$-monomorphism $\tau : \mathbb{F} \to \mathbb{D}$ where $\tau(a) = a$ for all $a \in \mathbb{F}$ and by taking $(\mathbb{A},\sigma) := (\mathbb{S},\sigma)$ with the σ-isomorphism $\rho : \mathbb{D} \to \mathbb{A}$ where $\rho(a) = a$ for all $a \in \mathbb{D}$.

Induction assumption: Otherwise, suppose that $1 \leq i < e$ and write $\mathbb{H}' := \mathbb{F}(t_1)\dots(t_{i-1})$. Assume that there is a $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{D},\sigma)$ with maximal depth d_{i-1} and a Σ^*-extension $(\mathbb{A},\sigma)$ of $(\mathbb{H}',\sigma)$ with $\mathbb{A} := \mathbb{H}'(s_1)\dots(s_r)$ together with an $\mathbb{F}$-monomorphism $\tau : \mathbb{H}' \to \mathbb{E}$ with $\delta(\tau(a)) \leq \delta(a)$ for all $a \in \mathbb{H}'$ and a σ-isomorphism $\rho : \mathbb{E} \to \mathbb{A}$ with $\rho(\tau(a)) = a$ for all $a \in \mathbb{H}'$.

Induction step: (I) First suppose that t_i is a Σ^*-extension. Define $f := \tau(\sigma(t_i) - t_i) \in \mathbb{E}$. Note that

$$\delta(f) \leq \delta(\sigma(t_i) - t_i) < d_i \qquad (4)$$

by assumption. **(I.i)** Suppose that there is no $g \in \mathbb{E}$ with $\sigma(g) - g = f$. Then we can construct the Σ^*-ext. $(\mathbb{E}(y),\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(y) = y + f$ by Thm. 1 and the $\mathbb{F}$-monomorphism $\tau' : \mathbb{H}'(t_i) \to \mathbb{E}$ with $\tau'(a) = \tau(a)$ for all $a \in \mathbb{H}'$ and $\tau'(t_i) = y$. With (4) we have $\delta(y) = \delta(f) + 1 \leq d_i$ and consequently, using our induction assumption, $\delta(\tau'(a)) \leq \delta(a)$ for all $a \in \mathbb{H}(t_i)$. Moreover, the Σ^*-ext. $(\mathbb{E}(y),\sigma)$ of $(\mathbb{D},\sigma)$ has maximal depth d_i. Furthermore, by Prop. 1.4 we can construct a Σ^*-ext. $(\mathbb{A}(t_i),\sigma)$ of $(\mathbb{A},\sigma)$ with the σ-isomorphism $\rho' : \mathbb{E}(y) \to \mathbb{A}(t_i)$ with $\rho'(a) = \rho(a)$ for all $a \in \mathbb{A}$ and $\rho'(y) = t_i$. Hence $\rho'(\tau(a)) = a$ for all $a \in \mathbb{H}'$ and $\rho'(\tau'(t_i)) = \rho'(y) = t_i$, i.e., $\rho'(\tau'(a)) = a$ for all $a \in \mathbb{H}'(t_i)$. By reordering we get a Σ^*-ext. $(\mathbb{A}',\sigma)$ of $(\mathbb{H}(t_i),\sigma)$ with our isomorphism $\rho' : \mathbb{E}(y) \to \mathbb{A}'$. This shows the induction step for this particular case. **(I.ii)** Suppose there is a $y \in \mathbb{E}$ with $\sigma(y) - y = f$. Since $(\mathbb{E},\sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{D},\sigma)$ with maximal depth $d_{i-1} \leq d_i$, we can apply Lemma 4 and obtain by reordering of $(\mathbb{E},\sigma)$ a δ-optimal ordered $\Pi\Sigma^*$-extension $(\mathbb{G}(z_1)\dots(z_l)(x_1)\dots(x_u),\sigma)$ of $(\mathbb{G}(z_1)\dots(z_l),\sigma)$ where $\delta(\mathbb{G}(z_1)\dots(z_l)) \leq d_i$ and $\delta(x_j) > d_i$. Hence with (4) we can apply Lemma 3.1 and get $\delta(y) \leq d_i$, i.e., $\delta(y) \leq \delta(t_i)$. In particular, we get the $\mathbb{F}$-monomorphism $\tau' : \mathbb{H}'(t_i) \to \mathbb{E}$ with $\tau'(a) = \tau(a)$ for all $a \in \mathbb{H}'$ and $\tau'(t_i) = y$. Then by the previous considerations and our induction assumption it follows that $\delta(\tau'(a)) \leq \delta(a)$ for all $a \in \mathbb{H}'(t_i)$. What remains to show is that there is a Σ^*-ext. $(\mathbb{A}',\sigma)$ of $(\mathbb{H}'(t_i),\sigma)$ with a σ-isomorphism $\rho' : \mathbb{E} \to \mathbb{A}'$ with $\rho'(\tau'(a)) = a$ for all $a \in \mathbb{H}'(t_i)$. Define $h := \rho(y) \in \mathbb{A}$. Then $\sigma(h) - h = \rho(f)$. Let j be minimal such that $h \in \mathbb{H}'(s_1)\dots(s_j)$. By Prop. 1.3, $(\mathbb{H}'(s_1)\dots(s_{j-1})(s_j),\sigma)$ and $(\mathbb{H}'(s_1)\dots(s_{j-1})(t_i),\sigma)$ are isomorphic with $\lambda(a) = a$ for all $a \in \mathbb{H}'(s_1)\dots(s_{j-1})$ and $\lambda(t_i) = h$. Hence the reordered $(\mathbb{H}'(t_i)(s_1)\dots(s_{j-1}),\sigma)$, $(\mathbb{H}'(s_1)\dots(s_{j-1})(s_j),\sigma)$ are isomorphic with λ. By Prop. 1.4 there is a Σ^*-ext. $(\mathbb{A}',\sigma)$ of $(\mathbb{H}'(t_i)(s_1)\dots(s_{j-1}),\sigma)$ which is isomorphic to $(\mathbb{A},\sigma)$ with $\lambda : \mathbb{A}' \to \mathbb{A}$ where $\lambda(a) = a$ for $a \in \mathbb{H}'$ and $\lambda(t_i) = h$. Take the σ-isomorphism $\rho' : \mathbb{E} \to \mathbb{A}'$ with $\rho'(a) := \lambda^{-1}(\rho(a))$ for all $a \in \mathbb{E}$. Then $\rho'(\tau'(a)) = \lambda^{-1}(\rho(\tau(a))) = \lambda^{-1}(a) = a$ for $a \in \mathbb{H}'$ and $\rho'(\tau'(t_i)) = \lambda^{-1}(\rho(y)) = \lambda^{-1}(h) = t_i$, i.e., $\rho'(\tau'(a)) = a$ for $a \in \mathbb{H}'(t_i)$.

(II) Suppose that t_i is a Π-ext., i.e., $\alpha := \sigma(t_i)/t_i \in \mathbb{H}'$. Moreover, assume that there is a $g \in \mathbb{E}$ and an $n > 0$ with $\sigma(g)/g = \tau(\alpha)^n$. Then there is a $g' \in \mathbb{A}$ with $\sigma(g')/g' = \rho(\tau(\alpha))^n = \alpha^n$. Let j be minimal s.t. $g' \in \mathbb{H}'(s_1)\dots(s_j)$. We have $j \geq 1$, since otherwise t_i is not a Π-ext. over $\mathbb{H}'$. Applying Lemma 1.1 shows that such a solution $g' \in \mathbb{H}'(s_1)\dots(s_j)$ does not exist, a contradiction. Therefore, there is a Π-extension $(\mathbb{E}(y),\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(y) = f\,y$ where $f := \tau(\alpha)$. Now we can follow the proof idea as in case (I.i) to complete the induction step. Namely, we construct the $\mathbb{F}$-monomorphism $\tau' : \mathbb{H}'(t_i) \to \mathbb{E}$ with $\tau'(a) = \tau(a)$ for all $a \in \mathbb{H}'$ and $\tau'(t_i) = y$. With $\delta(f) = \delta(\tau(\alpha)) \leq \delta(\alpha) < d_i$ we have $\delta(y) = \delta(f) + 1 \leq d_i$ and consequently, using our induction assumption, $\delta(\tau'(a)) \leq \delta(a)$ for all $a \in \mathbb{H}(t_i)$. Moreover, the Π-extension $(\mathbb{E}(y),\sigma)$ of $(\mathbb{D},\sigma)$ has maximal depth d_i. Furthermore, by Prop. 1.4 we can construct a Π-extension $(\mathbb{A}(t_i),\sigma)$ of $(\mathbb{A},\sigma)$ with the isomorphism $\rho' : \mathbb{E}(y) \to \mathbb{A}(t_i)$ with $\rho'(a) = \rho(a)$ for all $a \in \mathbb{A}$ and $\rho'(y) = t_i$. This means that $\rho'(\tau(a)) = a$ for all $a \in \mathbb{H}'$ and $\rho'(\tau'(t_i)) = \rho'(y) = t_i$, i.e., $\rho'(\tau'(a)) = a$ for all $a \in \mathbb{H}'(t_i)$. By reordering we get a Σ^*-extension $(\mathbb{A}',\sigma)$ of $(\mathbb{H}(t_i),\sigma)$ with our isomorphism $\rho' : \mathbb{E}(y) \to \mathbb{A}'$. This completes our inductive proof. $\square$

Finally, we explain how problem **B** and the problem to represent sums in δ-optimal extensions can be solved.

Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G},\sigma)$ with $\mathfrak{d} := \delta(\mathbb{F})$, $\boldsymbol{a} \in (\mathbb{F}^*)^2$, and $\mathbf{0} \neq \boldsymbol{f} \in \mathbb{F}^n$. $(\boldsymbol{a},\boldsymbol{f})$ is called $\mathbb{F}$-*complete*, if for any $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with maximal depth $\mathfrak{d}$ we have $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}) = \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$. We get immediately

Theorem 4. *Suppose $(\mathbb{F}(s),\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$ with $\sigma(s) = s + f$ and $\delta(s) = \delta(\mathbb{F}) + 1$. Then the extension s is δ-complete iff $((1,-1),(f))$ is $\mathbb{F}$-complete.*

Now the crucial observation is that problem **B** and the problem to represent sums can be reduced to problem

C: *Given* a δ-optimal ordered $\Pi\Sigma^*$-ext. $(\mathbb{F},\sigma)$ of $(\mathbb{G},\sigma)$, a homogenous $\boldsymbol{a} \in \mathbb{F}^2$ and $\boldsymbol{f} \in \mathbb{F}^n$. *Find* a Σ^*-ext. $(\mathbb{S},\sigma)$ of $(\mathbb{F},\sigma)$ where $(\mathbb{S},\sigma)$ is a δ-optimal ordered $\Pi\Sigma^*$-extension of $(\mathbb{G},\sigma)$ by reordering and where $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{S}$-complete.

• **Representing sums** in δ-optimal extensions: Suppose we have given a δ-optimal ordered $\Pi\Sigma^*$-extension $(\mathbb{F},\sigma)$ of $(\mathbb{G},\sigma)$ and given $f \in \mathbb{F}$. Then by solving **C** we obtain a $\Pi\Sigma^*$-extension $(\mathbb{S},\sigma)$ of $(\mathbb{F},\sigma)$ which can be reordered to a δ-optimal ordered $\Pi\Sigma^*$-extension of $(\mathbb{G},\sigma)$ and where $((1,-1),(f))$ is $\mathbb{S}$-complete. If there is a $g \in \mathbb{S}$ with $\sigma(g) - g = f$, we can represent the sum by $g \in \mathbb{S}$; see Exp. 6.2. By Lemma 6.3 this will always happen, if $\delta(f) < \delta(\mathbb{F})$.

Otherwise, if there is no such g and $\delta(\mathbb{F}) = \delta(f)$, take the Σ^*-extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s) - s = f$ and $\delta(s) = \delta(\mathbb{F}) + 1$. Then we can apply the following result.

Theorem 5. *Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G},\sigma)$, and let $(\mathbb{S},\sigma)$ be a Σ^*-extension of $(\mathbb{F},\sigma)$ which gives a δ-optimal ordered extension of $(\mathbb{G},\sigma)$ by reordering.*
Let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^2$ and $\boldsymbol{f} \in \mathbb{F}^n$. If $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{S}$-complete and $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{S}) = \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$, then $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{F}$-complete.

PROOF. Suppose $(\boldsymbol{a}, \boldsymbol{f})$ is not $\mathbb{F}$-complete, i.e., there is a $\Pi\Sigma^*$-extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}, \sigma)$ with maximal depth $\mathfrak{d} := \delta(\mathbb{F})$ and $g \in \mathbb{H} \setminus \mathbb{F}$, $\boldsymbol{c} \in \mathbb{K}^n$ s.t. $\sigma_{\boldsymbol{a}} g = \boldsymbol{c f}$. By Thm. 3 there is a $\Pi\Sigma^*$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{S}, \sigma)$ with maximal depth $\mathfrak{d}$ and an $\mathbb{F}$-monomorphism $\tau : \mathbb{H} \to \mathbb{E}$. Hence $\sigma_{\boldsymbol{a}} \tau(g) = \boldsymbol{c f}$. Since $(\boldsymbol{a}, \boldsymbol{f})$ is $\mathbb{S}$-complete, $\tau(g) \in \mathbb{S}$. By $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{S}) = \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$, $\tau(g) \in \mathbb{F}$; a contradiction. $\square$

Namely, by Theorem 5 $((1, -1), (f))$ is $\mathbb{F}$-complete. Hence by Theorem. 4 s is δ-optimal, and thus $(\mathbb{F}(s), \sigma)$ is a δ-optimal ordered $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$; see Example 4.

• **Solving B:** Suppose we have given a δ-optimal ordered $\Pi\Sigma^*$-extension $(\mathbb{F}, \sigma)$ of $(\mathbb{G}, \sigma)$ and given $\boldsymbol{f} \in \mathbb{F}^n$. Then Theorem 6 tells us how we can solve **B** by solving **C**.

THEOREM 6. *Let $(\mathbb{F}, \sigma)$, $(\mathbb{S}, \sigma)$ be as in Theorem 5. Suppose that $((1, -1), \boldsymbol{f})$ is $\mathbb{S}$-complete for $\boldsymbol{f} \in \mathbb{F}^n$. Then the following holds: If there is a solution g and $\boldsymbol{c} \in \mathbb{K}^n$ for problem* **B**, *there is also a $g \in \mathbb{S}$ with (1) and $\delta(g) = \delta(\boldsymbol{c f})$.*

PROOF. Let $(\mathbb{S}, \sigma)$ be such an extension and suppose that we have a solution of **B**, i.e., a $\Pi\Sigma^*$-extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathbb{H} = \mathbb{F}(x_1) \dots (x_r)$, a $\boldsymbol{c} \in \mathbb{K}^n$ and a $g \in \mathbb{H}^*$ with $\sigma(g) - g = \boldsymbol{c\,f} =: f$ and $\delta(g) = \delta(f)$. Hence $\delta(g) \leq \mathfrak{d} := \delta(\mathbb{F})$. Remove all x_i from $\mathbb{H}$ where $\delta(x_i) > \mathfrak{d}$. This gives a $\Pi\Sigma^*$-extension $(\mathbb{H}', \sigma)$ of $(\mathbb{S}, \sigma)$ with maximal depth $\mathfrak{d}$ where $g \in \mathbb{H}'$. By Thm. 3 there is a $\Pi\Sigma^*$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{S}, \sigma)$ with maximal depth $\mathfrak{d}$ and an $\mathbb{F}$-monomorphism $\tau : \mathbb{H}' \to \mathbb{E}$. Thus $\sigma(\tau(g)) - \tau(g) = f$ with $\tau(g) \in \mathbb{E}$. Since $((1, -1), \boldsymbol{f})$ is $\mathbb{S}$-complete, $\tau(g) \in \mathbb{S}$. $\square$

Remark: The two problems from above are closely related. Namely, if one represents sums in δ-optimal Σ^*-extensions as suggested above, we actually try to solve **B** with $\boldsymbol{f} = (f)$; see Exp. 6. Only if this fails, we construct a δ-optimal extension $(\mathbb{E}, \sigma)$ s.t. $g \in \mathbb{E}$ with $\sigma(g) - g = f$ and $\delta(g) = \delta(f) + 1$.

4. EXTENSION-STABLE REDUCTIONS

We sketch a reduction strategy presented in [9].

• With this reduction one can solve problem **A** if one can solve problem **A** in the ground field $(\mathbb{G}, \sigma)$ (see *Base case I*), and one can compute certain bounds (see *Boundings*).

• Afterwards we show some properties of this reduction in Lemma 5, which is the starting point for further refinements. Namely, we modify the reduction strategy in Section 5 (see Remark 1) which finally enables us to solve **C**.

Let $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{G}(t_1) \dots (t_e)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$, $\mathbb{K} := \mathrm{const}_\sigma \mathbb{G}$, $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{E}^2$ and $\boldsymbol{f} \in \mathbb{E}^n$.

The reduction strategy for $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{E})$:
If $a_1\, a_2 = 0$, a basis is immediate. Hence suppose $\boldsymbol{a} \in (\mathbb{E}^*)^2$.
Base case I: If $e = 0$, take a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{G})$.
Denote $\mathbb{H} := \mathbb{G}(t_1) \dots (t_{e-1})$, $t := t_e$; suppose $\sigma(t) = \alpha\, t + \beta$.
Boundings: First a *denominator bound* is needed, i.e., a $d \in \mathbb{H}[t]^*$ such that for all $\boldsymbol{c} \in \mathbb{K}^n$ and $g \in \mathbb{H}(t)$ with $\sigma_{\boldsymbol{a}} g = \boldsymbol{c f}$ we have $d\, g \in \mathbb{H}[t]$. Given such a d, define $\boldsymbol{a'} = (a'_1, a'_2) := (a_1/\sigma(d), a_2/d)\, q \in \mathbb{H}[t]^2$ and $\boldsymbol{f'} := \boldsymbol{f}\, q \in \mathbb{H}[t]^n$ for some $q \in \mathbb{H}(t)^*$; more precisely, take a q such that the denominators are cleared and common factors are cancelled in $\boldsymbol{a'}$ and $\boldsymbol{f'}$. Since $\{(\kappa_{i1}, \dots, \kappa_{in}, p_i)\}_{1 \leq i \leq \mu}$ is a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{H}[t])$ iff $\{(\kappa_{i1}, \dots, \kappa_{in}, \frac{p_i}{d})\}_{1 \leq i \leq \mu}$ is a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{H}(t))$, it suffices to find a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{H}[t])$. Next, need a *degree bound* $b \in \mathbb{N}_0 \cup \{-1\}$ for the polynomial solutions, i.e., a b s.t. $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{H}[t]) = \mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{H}[t]_b)$ and $\boldsymbol{f'} \in \mathbb{H}[t]_{l+b}$ where $l := \max(\deg(a'_1), \deg(a'_2)) \geq 0$.
Set $\delta := b$ and $\boldsymbol{f_\delta} := \boldsymbol{f'}$. Then we go on with the
Incremental reduction for $(\boldsymbol{a'}, \boldsymbol{f_\delta})$: Suppose that $\boldsymbol{a'} = (a'_1, a'_2) \in (\mathbb{H}[t]^*)^2$ with $l := \max(\deg(a'_1), \deg(a'_2)) \geq 0$ and $\boldsymbol{f_\delta} = (f_1, \dots, f_n) \in \mathbb{H}[t]^n_{\delta+l}$ for some $\delta \in \mathbb{N}_0 \cup \{-1\}$; as given from above. Then we look for all solutions $g = \sum_{i=0}^{\delta} g_i t^i \in \mathbb{H}[t]_\delta$ and $\boldsymbol{c} \in \mathbb{K}^n$ with $\sigma_{\boldsymbol{a_\delta}} g = \boldsymbol{c f_\delta}$ as follows. First derive the possible leading coefficients g_δ in $(\mathbb{H}, \sigma)$, then plug in the resulting solutions into $\sigma_{\boldsymbol{a'}} g = \boldsymbol{c f_\delta}$ and look for the remaining $g = \sum_{i=0}^{b-1} g_i t^i$ by recursion. More precisely, define

$$\boldsymbol{\tilde{a}_\delta} := ([a'_1]_l\, \alpha^\delta, [a'_2]_l) \text{ and } \boldsymbol{\tilde{f}_\delta} := ([f_1]_{\delta+l}, \dots, [f_n]_{\delta+l}) \quad (5)$$

where $\boldsymbol{0} \neq \boldsymbol{\tilde{a}_\delta} \in \mathbb{H}^2$ and $\boldsymbol{\tilde{f}_\delta} \in \mathbb{H}^n$; $[p]_l$ gives the l-th coefficient of $p \in \mathbb{H}[t]$. Afterwards the task is to find a basis $B_1 = \{(c_{i1}, \dots, c_{in}, w_i)\}_{1 \leq i \leq \lambda}$ of $\mathrm{V}(\boldsymbol{\tilde{a}_\delta}, \boldsymbol{\tilde{f}_\delta}, \mathbb{H})$. We follow

Reduction I: Apply *Reduction strategy* for $(\boldsymbol{\tilde{a}_\delta}, \boldsymbol{\tilde{f}_\delta}, \mathbb{H})$.

If $B_1 = \{\}$ then $\boldsymbol{c} = \boldsymbol{0}$ and $g \in \mathbb{H}[t]_{\delta-1}$ are the only choices for $\sigma_{\boldsymbol{a'}} g = \boldsymbol{c f_\delta}$. Hence, try to find a basis B_2 of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_{\delta-1}}, \mathbb{H}[t]_{\delta-1})$ with $\boldsymbol{f_{\delta-1}} := (0)$. Then the basis B_1 can be reconstructed. Otherwise, if $B_1 \neq \{\}$, define $\boldsymbol{C} := (c_{ij}) \in \mathbb{K}^{\lambda \times n}, \boldsymbol{g} := (w_1\, t^\delta, \dots, w_\lambda\, t^\delta) \in t^\delta\, \mathbb{H}^\lambda$ and

$$\boldsymbol{f_{\delta-1}} := \boldsymbol{C}\, \boldsymbol{f_\delta} - \sigma_{\boldsymbol{a'}} \boldsymbol{g}. \quad (6)$$

By construction, $\boldsymbol{f_{\delta-1}} \in \mathbb{H}[t]^\lambda_{\delta+l-1}$. Now we proceed as follows. Find all $h \in \mathbb{H}[t]_{\delta-1}$ and $\boldsymbol{d} \in \mathbb{K}^\lambda$ with $\sigma_{\boldsymbol{a'}}(h + \boldsymbol{d\, g}) = \boldsymbol{d\, C\, f_\delta}$ which is equivalent to $\sigma_{\boldsymbol{a'}} h = \boldsymbol{d\, f_{\delta-1}}$, i.e., find a basis B_2 of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_{\delta-1}}, \mathbb{H}[t]_{\delta-1})$. Then given B_1 and B_2, a basis for $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{H}[t]_\delta)$ can be computed; for more details see [9]. To get B_2, we follow

Reduction II: Apply *Incr. Reduction* for $(\boldsymbol{a'}, \boldsymbol{f_{\delta-1}})$.

If $\delta = -1$, we have reduced the problem to linear algebra.
Base case II: Take a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f_{-1}}, \{0\})$ which equals to $\{\boldsymbol{k} \in \mathbb{K}^n \mid \boldsymbol{f_{-1}\, k} = 0\} \times \{0\}$.

We call $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{H}(t))$ the *reduction problem* of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{H}(t))$.

• Following *Reduction II* and *Base case II* one gets an *incremental reduction* of $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{H}(t))$. The *incremental problems* are $((\boldsymbol{a'}, \boldsymbol{f_\delta}), \dots, (\boldsymbol{a'}, \boldsymbol{f_{-1}}))$ and the *coefficient problems* are $((\boldsymbol{\tilde{a}_\delta}, \boldsymbol{\tilde{f}_\delta}), \dots, (\boldsymbol{\tilde{a}_0}, \boldsymbol{\tilde{f}_0}))$. $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{H}(t))$ is called the *father-problem* of $(\boldsymbol{\tilde{a}_i}, \boldsymbol{\tilde{f}_i})$ for $0 \leq i \leq \delta$.

• Following *Reduction I* and *Base case I* one gets a tree of recursive reductions called a *reduction of* $(\boldsymbol{a}, \boldsymbol{f}, \mathbb{H}(t))$ *to* $\mathbb{G}$. We call a coefficient problem in the ground field $(\mathbb{G}, \sigma)$ within this reduction a $\mathbb{G}$*-problem.* A $\mathbb{G}$-problem $(\boldsymbol{a'}, \boldsymbol{f'})$ is *critical* if $\boldsymbol{a'}$ is homogeneous over $\mathbb{G}$ and if for all its father-problems $(\boldsymbol{a_i}, \boldsymbol{f_i}, \mathbb{G}(t_1) \dots (t_i))$ with $1 \leq i \leq e$ within the reduction the $\boldsymbol{a_i}$ are homogeneous over $\mathbb{G}(t_1) \dots (t_i)$.

Next, we introduce reductions to $\mathbb{F}$ that are extension-stable. A denominator bound $d \in \mathbb{H}[t_e]^*$ of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{H}(t_e))$ or a degree bound b of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{H}[t_e])$ is *extension-stable over* $\mathbb{G}$ if $\boldsymbol{a}$ is inhomogeneous over $\mathbb{H}(t_e)$ or the following holds: For any $\Pi\Sigma^*$-extension $(\mathbb{H}(t_e)(x_1) \dots (x_r), \sigma)$ of $(\mathbb{H}(t_e), \sigma)$ that can be reordered to the $\Pi\Sigma^*$-extension $(\mathbb{F}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{F} := \mathbb{G}(x_1) \dots (x_r)(t_1) \dots (t_e)$ the polynomial d is a denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$. Similarly, b must also be a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{G}(x_1) \dots (x_r)(t_1) \dots (t_{e-1})[t_e])$.
We call a reduction of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{H}(t_e))$ to $\mathbb{G}$ *extension-stable* if all denominator and degree bounds within the reduction to $\mathbb{G}$ are extension-stable over $\mathbb{G}$.
Finally, we say that $(\mathbb{G}, \sigma)$ is *depth-computable*, in short δ-computable, if one can handle base case I for any $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{G}^2$, $\boldsymbol{f} \in \mathbb{G}^n$, and for any $\Pi\Sigma^*$-extension $(\mathbb{H}(t), \sigma)$ of

$(\mathbb{G},\sigma)$ the following holds: one can hanlde base case II for $\boldsymbol{f_{-1}}\in\mathbb{H}[t]^n$, and one can compute extension stable denominator and degree bounds. *Given these properties a basis of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$*) can be computed by our reduction.*

By the results in [6, Thm. 8.2], [4, Thm. 7.3] it follows that such extension-stable degree bounds exist. Moreover, they can be computed if $(\mathbb{G},\sigma)$ is a $\Pi\Sigma^*$-field over a σ-computable constant field $\mathbb{K}$. In this case *Base case II* can be solved by linear algebra methods and *Base case I* can be handled by applying our reduction again. Summarizing, we obtain

THEOREM 7. *Let* $(\mathbb{E},\sigma)$ *with* $\mathbb{E}:=\mathbb{G}(t_1)\dots(t_e)$ *be a* $\Pi\Sigma^*$*-extension of* $(\mathbb{G},\sigma)$*.* **(1)** *There is an extension-stable reduction of* $(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t_e))$ *to* $\mathbb{G}$ *for* $\boldsymbol{0}\neq\boldsymbol{a}\in\mathbb{E}^2$ *and* $\boldsymbol{f}\in\mathbb{E}^n$*.* **(2)** *A* $\Pi\Sigma^*$*-field* $(\mathbb{G},\sigma)$ *over a* σ*-computable* $\mathbb{K}$ *is* δ*-computable.*

Example 3. In the $\Pi\Sigma^*$-field from Exp. 1.1 there is the following extension-stable reduction $(\boldsymbol{a},\boldsymbol{f},\mathbb{Q}(t_1)(t_2))$ to $\mathbb{Q}(t_1)$ for $\boldsymbol{a}=(1,-1)$, $\boldsymbol{f}=\sigma(t_2/t_1^2)$. Take for $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{Q}(t_1)(t_2))$ the extension-stable denominator 1, for $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{Q}(t_1)[t_2])$ the extension-stable degree bound 2. We get the coefficient problems $(((t_1+1)^2,-(t_1+1)^2),\tilde{\boldsymbol{f}}_{\boldsymbol{i}})$ with $\tilde{\boldsymbol{f}}_{\boldsymbol{2}}=(0)$, $\tilde{\boldsymbol{f}}_{\boldsymbol{1}}=(-2(t_1+1),1)$ and $\tilde{\boldsymbol{f}}_{\boldsymbol{0}}=(t_1+1)$; these are the $\mathbb{Q}(t_1)$-critical problems in our reduction.

Finally, we show the following generalization of [8, Prop. 1].

LEMMA 5. *Let* $(\mathbb{E}(x),\sigma)$ *be a* $\Pi\Sigma^*$*-extension of* $(\mathbb{F},\sigma)$ *with* $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ *and* $\sigma(x)=\alpha\,x+\beta$ *where* $\alpha,\beta\in\mathbb{F}$*; consider the reordered* $\Pi\Sigma^*$*-extension* $(\mathbb{F}(x)(t_1)\dots(t_e),\sigma)$ *of* $(\mathbb{F},\sigma)$*. Let* $\boldsymbol{a}\in\mathbb{E}^2$ *be homogeneous over* $\mathbb{E}$*,* $\boldsymbol{f}\in\mathbb{E}^n$ *and take an extension-stable reduction of* $(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$ *to* $\mathbb{F}$ *where* S *contains all* $\mathbb{F}$*-critical problems. If for all* $(\boldsymbol{a'},\boldsymbol{f'})\in S$ *we have* $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F})=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(x))$ *then* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(x))$*. Moreover, there is an extension-stable reduction of* $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(x)(t_1)\dots(t_e))$ *to* $\mathbb{F}(x)$ *where all the* $\mathbb{F}(x)$*-critical problems are given by* S*.*

PROOF. The proof will be done by induction on the number of extensions. If $e=0$, nothing has to be shown. Otherwise suppose that the lemma holds for the first $e-1$ extensions with $e\geq 1$. Let $(\mathbb{F}(t_1)\dots(t_e)(x),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\sigma(x)=\alpha\,x+\beta$, $\alpha,\beta\in\mathbb{F}$, and consider the reordered $\Pi\Sigma^*$-extension $(\mathbb{F}(x)(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$. Denote $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_{e-1})$, $t:=t_e$ and $\mathbb{H}:=\mathbb{F}(x)(t_1)\dots(t_{e-1})$ as shortcut. Let $\boldsymbol{a}\in\mathbb{E}(t)^2$ be homogeneous over $\mathbb{E}(t)$, $\boldsymbol{f}\in\mathbb{E}(t)^n$, take an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t))$ to $\mathbb{F}$ where S contains all $\mathbb{F}$-critical problems, and suppose that $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F})=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(x))$ for all $(\boldsymbol{a'},\boldsymbol{f'})\in S$. Then we show that $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t))=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t))$. Moreover, as a by-product, we show that there is an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t))$ to $\mathbb{F}(x)$ with the $\mathbb{F}(x)$-critical problems given by S.
In the extension-stable reduction let $d\in\mathbb{E}[t]^*$ be the denominator bound of the solution space $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t))$. Since $\boldsymbol{a}$ is homogeneous over $\mathbb{E}(t)$, $d\in\mathbb{H}[t]$ is also a denominator bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t))$; by definition it is extension-stable. After clearing denominators and cancelling common factors, we get $\boldsymbol{a'}=(a'_1,a'_2):=(a_1/\sigma(d),a_2/d)\,q\in\mathbb{E}[t]^2$ and $\boldsymbol{f'}:=\boldsymbol{f}\,q\in\mathbb{E}[t]^n$ for some $q\in\mathbb{E}(t)^*$ in our reduction. Note that $\boldsymbol{a'}$ is still homogeneous over $\mathbb{E}(t)$: we have $\sigma_{\boldsymbol{a'}}h'=0$ with $h':=h\,d\in\mathbb{H}[t]^*$ for some $h\in\mathbb{E}(t)^*$ with $\sigma_{\boldsymbol{a}}h=0$. Now it suffices to show that $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{H}[t])=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{E}[t])$. In the given reduction let b be the extension-stable degree bound of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{E}[t])$. Therefore b is a degree bound of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{H}[t])$; it is also extension-stable. Hence, we have to show $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{E}[t]_b)=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{H}[t]_b)$. Let $((\boldsymbol{a'},\boldsymbol{f_b}),\dots,(\boldsymbol{a'},\boldsymbol{f_{-1}}))$ be the incremental problems and $((\tilde{\boldsymbol{a}}_{\boldsymbol{b}},\tilde{\boldsymbol{f}}_{\boldsymbol{b}}),\dots,(\tilde{\boldsymbol{a}}_{\boldsymbol{0}},\tilde{\boldsymbol{f}}_{\boldsymbol{0}}))$ be the coefficient-problems in the incremental reduction. We show $\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{E})=\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{H})$ for all $0\leq i\leq b$. First suppose that $\tilde{\boldsymbol{a}}_{\boldsymbol{i}}$ is inhomogeneous over $\mathbb{E}$. Note that $\tilde{\boldsymbol{a}}_{\boldsymbol{i}}=([a'_1]_l\,\alpha^i,[a'_2]_l)$ by (5). Since $\sigma_{\boldsymbol{a'}}h'=0$, we get by coefficient comparison $\alpha^k[a'_1]_l\sigma(h'')+[a'_2]_lh''=0$ where $k:=\deg(h')$ and $h''\in\mathbb{E}^*$ is the leading coefficient of h'. Hence $(\alpha^k[a'_1]_l,[a'_2]_l)\in(\mathbb{E}^*)^2$ is homogeneous over $\mathbb{E}$. Since $\tilde{\boldsymbol{a}}_{\boldsymbol{i}}$ is inhomogeneous, $i\neq k$ and $\alpha\neq 1$, i.e., t is a Π-extension. Therefore, by Lemma 1.2 $\tilde{\boldsymbol{a}}_{\boldsymbol{i}}$ is inhomogeneous over $\mathbb{E}(x)$ and $\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{E})=\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{E}(x))$. Thus $\tilde{\boldsymbol{a}}_{\boldsymbol{i}}$ is inhomogeneous over $\mathbb{H}$ and $\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{E})=\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{H})$. In particular, there are no $\mathbb{F}$-critical problems in $(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{E})$ to $\mathbb{F}$ and no $\mathbb{F}(x)$-critical problems in $(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{H})$ to $\mathbb{F}(x)$. Otherwise, assume that $\tilde{\boldsymbol{a}}_{\boldsymbol{i}}$ is homogeneous over $\mathbb{E}$. Then the extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t))$ to $\mathbb{F}$ contains an extension-stable reduction of $(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{E})$ to $\mathbb{F}$ and all the $\mathbb{F}$-critical problems of the reduction of $(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{E})$ are given by a subset S_i of S. Hence with the induction assumption it follows that $\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{E})=\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{H})$ and the $\mathbb{F}(x)$-critical problem in $(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{H})$ to $\mathbb{F}(x)$ are also S_i. Since $\mathbb{E}[t]_{-1}=\mathbb{H}[t]_{-1}=\{0\}$, $\mathrm{V}(\boldsymbol{a},\boldsymbol{f_{-1}},\mathbb{E}[t]_{-1})=\mathrm{V}(\boldsymbol{a},\boldsymbol{f_{-1}},\mathbb{H}[t]_{-1})$. Thus, we get an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t))$ to $\mathbb{F}(x)$ where the $\mathbb{F}(x)$-critical problems are given by S. By construction, $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f_i},\mathbb{E}[t]_i)=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f_i},\mathbb{H}[t]_i)$ for all i. Hence $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t))=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t))=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t)(x))$. ☐

5. SOLVING PROBLEM C

We will solve **C** (Theorem 8) by refining the reduction from above. Some special cases (Lemma 6) are immediate.

LEMMA 6. *Let* $(\mathbb{F},\sigma)$ *be a* $\Pi\Sigma^*$*-ext. of* $(\mathbb{G},\sigma)$ *with* $\mathfrak{d}:=\delta(\mathbb{F})$*,* $\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}^2$ *be homogeneous,* $\boldsymbol{f}\in\mathbb{F}^n$ *and* $\mathbb{V}:=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$*.* **(1)** *If* $\mathfrak{d}=0$ *or* $\dim\mathbb{V}=n+1$*, then* $(\boldsymbol{a},\boldsymbol{f})$ *is* $\mathbb{F}$*-complete.* **(2)** *If* $\mathfrak{d}=1$*,* $\mathrm{const}_\sigma\mathbb{G}=\mathbb{G}$ *and* $\sigma(g)-g\in\mathbb{G}^*$ *for some* $g\in\mathbb{F}$*, then* $(\boldsymbol{a},\boldsymbol{f})$ *is* $\mathbb{F}$*-complete.* **(3)** *If* $\delta(\boldsymbol{f}),\delta(\boldsymbol{a})<\delta(\mathbb{F})$ *and* $(\boldsymbol{a},\boldsymbol{f})$ *is* $\mathbb{F}$*-complete, then* $\dim\mathbb{V}=n+1$*.*

PROOF. **(1)** is obvious. **(2)** Suppose $(\boldsymbol{a},\boldsymbol{f})$ is not $\mathbb{F}$-complete, i.e., there is a $\Pi\Sigma^*$-ext. $\mathbb{D}:=\mathbb{F}(x_1)\dots(x_r)$ with depth 1, $h\in\mathbb{D}\setminus\mathbb{F}$ and $\boldsymbol{c}\in\mathbb{K}^n$ s.t. $\sigma_{\boldsymbol{a}}h=\boldsymbol{c}\boldsymbol{f}$. By Prop. 1.2 there is an x_i with $\sigma(x_i)-x_i=:k\in\mathbb{G}^*$. Hence, $\sigma(c\,g)-c\,g=k$ with $c:=k/(\sigma(g)-g)\in\mathbb{G}^*$, a contradiction that x_i is a Σ^*-ext. by Thm. 1. **(3)**. Suppose $\dim\mathbb{V}<n+1$, i.e., there is a $\boldsymbol{c}\in\mathbb{K}^n$ s.t. there is no $g\in\mathbb{F}$ with $\sigma_{\boldsymbol{a}}g=\boldsymbol{c}\boldsymbol{f}=:f$. Take $h\in\mathbb{F}^*$ with $\sigma_{\boldsymbol{a}}h=0$. Then there is no $g\in\mathbb{F}$ with $\sigma(g)-g=-f/(ha_2)$ where $\boldsymbol{a}=(a_1,a_2)$. Thus there is the Σ^*-ext. $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s)=s-f/(ha_2)$, $\delta(s)\leq\delta(\mathbb{F})$ and $\sigma_{\boldsymbol{a}}s=f$. Hence $(\boldsymbol{a},\boldsymbol{f})$ is not $\mathbb{F}$-complete. ☐

THEOREM 8. *Let* $(\mathbb{F},\sigma)$ *be a* δ*-optimal ordered* $\Pi\Sigma^*$*-ext. of* $(\mathbb{G},\sigma)$*,* $\boldsymbol{a}\in\mathbb{F}^2$ *be homogeneous and* $\boldsymbol{f}\in\mathbb{F}^n$*. Then there is a* Σ^**-extension* $(\mathbb{S},\sigma)$ *of* $(\mathbb{F},\sigma)$ *where* $(\mathbb{S},\sigma)$ *is a* δ*-optimal ordered* $\Pi\Sigma^*$*-ext. of* $(\mathbb{G},\sigma)$ *by reordering and where* $(\boldsymbol{a},\boldsymbol{f})$ *is* $\mathbb{S}$*-complete. It can be computed if* $(\mathbb{G},\sigma)$ *is* δ*-computable.*

We proceed as follows. Using Lemma 5 from Section 4 we provide a sufficient condition (Condition A) in Proposition 2 that guarantees that the solution space cannot be increased by extensions with maximal depth $\mathfrak{d}-1$. Given this result we can derive a criterion wether $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{S}$-complete for a given Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{E},\sigma)$; see Thm. 9.

Condition A: Let $(\mathbb{E},\sigma)$ with $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ where $\delta(\mathbb{F})=\mathfrak{d}-1$ and $\delta(t_i)\geq\mathfrak{d}$. Let $\boldsymbol{a}\in\mathbb{E}^2$ be homogeneous over $\mathbb{E}$ and $\boldsymbol{f}\in\mathbb{E}^n$, and suppose that all $\mathbb{F}$-critical problems, say $S=\{(\boldsymbol{a_i},\boldsymbol{f_i})\}_{1\leq i\leq k}$ with $\boldsymbol{a_i}=(a_{i1},a_{i2})$, $\boldsymbol{f_i}=(f_{i1},\dots,f_{ir_i})\in\mathbb{F}^{r_i}$, of an extension-stable reduction of $\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E})$ to $\mathbb{F}$ are $\mathbb{F}$-complete.

Proposition 2. *Suppose that Condition A holds, and let $(\mathbb{S},\sigma)$ with $\mathbb{S}=\mathbb{E}(x_1)\dots(x_r)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{E},\sigma)$ with maximal depth $\mathfrak{d}-1$. Then $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{S})$. Moreover, for the reordered difference field $(\mathbb{H}(t_1)\dots(t_e),\sigma)$ with $\mathbb{H}=\mathbb{F}(x_1)\dots(x_r)$ there exists an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t_1)\dots(t_e))$ to $\mathbb{H}$ with the $\mathbb{H}$-critical problems S which are all $\mathbb{H}$-complete.*

Proof. Since all $\mathbb{F}$-critical problems are $\mathbb{F}$-complete, we have $\mathrm{V}(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F})=\mathrm{V}(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F}(x_1))=\dots=\mathrm{V}(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{H})$. By applying Lemma 5 r times, it follows that there is an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t_1)\dots(t_e))$ to $\mathbb{H}$ with the $\mathbb{H}$-critical problems given by S; clearly they are $\mathbb{H}$-complete. Moreover, $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{S})$. $\square$

Theorem 9. *Suppose that Cond. A holds with $\delta(t_i)=\mathfrak{d}$. If $(\mathbb{S},\sigma)$ is a Σ^*-extension of $(\mathbb{E},\sigma)$ with maximal depth $\mathfrak{d}$ where for any $1\leq i\leq k$ and $1\leq j\leq r_i$ there is a $g\in\mathbb{D}^*$ with $a_{i1}\,\sigma(g)-a_{i2}\,g=f_{ij}$ then $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{S}$-complete.*

Proof. Suppose that $(\boldsymbol{a},\boldsymbol{f})$ is not $\mathbb{S}$-complete, i.e., there is a $\Pi\Sigma^*$-ext. $(\mathbb{H},\sigma)$ of $(\mathbb{S},\sigma)$ with maximal depth $m\leq\mathfrak{d}$, a $g\in\mathbb{H}\setminus\mathbb{S}$ and $\boldsymbol{c}\in\mathbb{K}^n$ with $\sigma_{\boldsymbol{a}}g=\boldsymbol{c}\boldsymbol{f}$. Let m be minimal. By [8, Lemma 1] we may refine this assumption to $\mathbb{H}=\tilde{\mathbb{H}}(s)$ with $\delta(s)=m$ and $\sigma(s)-s\in\tilde{\mathbb{H}}$ where $(\tilde{\mathbb{H}},\sigma)$ is a $\Pi\Sigma^*$-ext. of $(\mathbb{S},\sigma)$ with maximal depth $m-1$ and $g\in\tilde{\mathbb{H}}(s)\setminus\tilde{\mathbb{H}}$. Subsequently, write $\tilde{\mathbb{H}}=\mathbb{E}(x_1)\dots(x_r)$ with $\delta(x_i)<\mathfrak{d}$. Now consider the extension-stable reduction as claimed above and take the reordered $\Pi\Sigma^*$-ext. $(\mathbb{F}(x_1)\dots(x_r)(t_1)\dots(t_r),\sigma)$ of $(\mathbb{F},\sigma)$; denote $\tilde{\mathbb{F}}:=\mathbb{F}(x_1)\dots(x_r)$. Applying Prop. 2 we get an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\tilde{\mathbb{F}}(t_1)\dots(t_e))$ to $\tilde{\mathbb{F}}$ with the $\tilde{\mathbb{F}}$-critical problems S which are all $\tilde{\mathbb{F}}$-complete. By Lemma 5 together with $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})\subsetneq\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H})$ it follows that there is an $(\boldsymbol{a'},\boldsymbol{f'})\in S$ with $\boldsymbol{f'}\in\mathbb{F}^\nu$ and $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F})=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\tilde{\mathbb{F}})\subsetneq\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\tilde{\mathbb{F}}(s))$. Therefore, there is a $g'\in\tilde{\mathbb{F}}(s)\setminus\tilde{\mathbb{F}}$ and $\boldsymbol{c'}\in\mathbb{K}^\nu$ with $\sigma_{\boldsymbol{a'}}g'=\boldsymbol{c'}\boldsymbol{f'}$. In particular, $g'\in\tilde{\mathbb{H}}(s)\setminus\mathbb{S}$. By assumption on the Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{E},\sigma)$ there are $g_i\in\mathbb{S}$ with $\sigma_{\boldsymbol{a'}}g_i=f'_i$. Hence for $h':=\boldsymbol{c}\,\boldsymbol{h}\in\mathbb{S}$ with $\boldsymbol{h}=(g_1,\dots,g_\nu)$ we have $\sigma_{\boldsymbol{a'}}h'=\boldsymbol{c'}\boldsymbol{f'}$, a contradiction to Prop. 1.1. Hence $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{S}$-complete. $\square$

Example 4. With this result and Theorems 4 and 5 we can test if the extension t_3 in Exp. 1.1 is δ-optimal: Take the reduction to $\mathbb{Q}(t_1)$ from Exp. 3. By Lemma 6.2 the $\mathbb{Q}(t_1)$-critical problems are $\mathbb{Q}(t_1)$-complete, i.e., Condition A holds. Take $(\mathbb{Q}(t_1)(t_2)(x'_1),\sigma)$ with $\sigma(x'_1)=x'_1+1/(t_1+1)^2$. Since there are $g\in\mathbb{Q}(t_1)(t_2)(x'_1)$ with $(t_1+1)^2\sigma(g)-(t_1+1)^2g=f$ for $f\in\{0,-2(t_1+1),1,1/(t_1+1)\}$, $P:=((1,-1),(\sigma(t_2/t_1^2)))$ is $\mathbb{Q}(t_1)(t_2)(x'_1)$-complete by Thm. 9. Since there is no $g\in\mathbb{Q}(t_1)(t_2)(x'_1)$ with $\sigma(g)-g=\sigma(t_2/t_1^2)$, P is $\mathbb{Q}(t_1)(t_2)$-complete by Thm. 5. Thus t_3 is δ-complete by Thm. 4.

Finally, we prove Thm. 8 by showing that such an extension $(\mathbb{S},\sigma)$ supposed in Thm. 9 exists. More precisely, in Lemma 7 we show how we can construct an extension s.t. Condition *A* holds (see Alg. 1), and in Lemma 8 we show how we can construct an extension $(\mathbb{S},\sigma)$ with the criterion in Thm. 9 (see Alg. 2). The resulting algorithms are applicable if $(\mathbb{G},\sigma)$ is δ-computable.

The corresponding proofs are done inductively/recursively: under the assumption that Theorem 8 holds for the depth level $\mathfrak{d}-1$ we show the desired results for the depth level $\mathfrak{d}$.

Lemma 7. *Suppose that Thm. 8 holds with the restriction that $\delta(\mathbb{F})=\mathfrak{d}-1$. Let $(\mathbb{E},\sigma)$ be a δ-optimal ordered $\Pi\Sigma^*$-extension of $(\mathbb{G},\sigma)$ where $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ with $\delta(\mathbb{F})=\mathfrak{d}-1$ and $\delta(t_i)=\mathfrak{d}$; let $\boldsymbol{a}\in\mathbb{E}^2$ be homogeneous over $\mathbb{E}$ and $\boldsymbol{f}\in\mathbb{E}^n$. Then there is a Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{E},\sigma)$ with maximal depth $\mathfrak{d}-1$ that can be reordered to a δ-optimal ordered $\Pi\Sigma^*$-extension $(\mathbb{D}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{G},\sigma)$ with $\delta(\mathbb{D})=\mathfrak{d}-1$ such that the following holds: there is an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{D}(t_1)\dots(t_e))$ to $\mathbb{D}$ where all $\mathbb{D}$-critical problems are $\mathbb{D}$-complete. If $(\mathbb{G},\sigma)$ is δ-computable, such an extension can be computed.*

Proof. If $e=0$, the lemma follows by using the depth-restricted version of Thm. 8. Otherwise suppose that the lemma holds for a $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{H}:=\mathbb{F}(t_1)\dots(t_{e-1})$, $e\geq 1$. Now take a δ-optimal $\Pi\Sigma^*$-ext. $(\mathbb{H}(t),\sigma)$ of $(\mathbb{H},\sigma)$ with $\delta(t)=\mathfrak{d}$; let $\boldsymbol{f}\in\mathbb{H}(t)^n$ and $\boldsymbol{a}\in\mathbb{H}(t)^2$ be homogeneous. Then we show that the lemma holds for $(\mathbb{H}(t),\sigma)$. Take an extension-stable denominator bound $d\in\mathbb{H}[t]^*$ of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t))$. Set $\boldsymbol{a'}:=(a_1/\sigma(d),a_2/d)\in\mathbb{H}(t)^2$, $\boldsymbol{f'}:=\boldsymbol{f}$ and clear denominators and common factors s.t. $\boldsymbol{a'}\in(\mathbb{H}[t]^*)^2$ and $\boldsymbol{f'}\in\mathbb{H}[t]^n$. Take an extension-stable degree bound b of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{H}[t])$. Now we show that there is a Σ^*-ext. $(\mathbb{S},\sigma)$ of $(\mathbb{H}(t),\sigma)$ with maximal depth $\mathfrak{d}-1$ that can be reordered to a δ-optimal ordered $\Pi\Sigma^*$-extension $(\mathbb{D}(t_1)\dots(t_{e-1})(t),\sigma)$ of $(\mathbb{G},\sigma)$ with $\delta(\mathbb{D})=\mathfrak{d}-1$ such that for all coefficient problems there is an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{D}(t_1)\dots(t_{e-1}))$ to $\mathbb{D}$ in which all $\mathbb{D}$-critical problems are $\mathbb{D}$-complete. If $b=-1$, nothing has to be shown. Otherwise, suppose that we have obtained such an extension that gives extension-stable reductions for the first $u\geq 0$ coefficient problems in which all $\mathbb{D}$-problems are $\mathbb{D}$-complete. Denote $\mathbb{B}:=\mathbb{D}(t_1)\dots(t_{e-1})$ and let $(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{f}})$ be the $u+1$-th coefficient problem. If $\tilde{\boldsymbol{a}}$ is inhomogeneous, no additional $\mathbb{D}$-critical problems appear. Hence our extension gives extension-stable reductions for the first $u+1$ coefficient problems. Otherwise, if $\tilde{\boldsymbol{a}}$ is homogeneous, we can apply our induction assumption and get a Σ^*-extension $(\mathbb{S}',\sigma)$ of $(\mathbb{B},\sigma)$ with maximal depth $\mathfrak{d}-1$, i.e., $\mathbb{S}'=\mathbb{B}(s_1)\dots(s_r)$ with $\delta(s_i)<\mathfrak{d}$ and with the following properties. We can reorder the extension to a δ-optimal ordered $\Pi\Sigma^*$-extension $(\mathbb{B}',\sigma)$ of $(\mathbb{G},\sigma)$ with $\mathbb{B}':=\mathbb{D}'(t_1)\dots(t_{e-1})$ and $\delta(\mathbb{D}')<\mathfrak{d}$ such that all $\mathbb{D}'$-critical problems in $(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{f}},\mathbb{B}')$ to $\mathbb{D}'$ are $\mathbb{D}'$-complete. Then we apply Prop. 2 and it follows that also all the $\mathbb{D}'$-critical problems in the extension-stable reductions of the first u coefficient-problems are $\mathbb{D}'$-complete. In particular, the corresponding solution spaces are the same. Hence, we obtain an extension where the first $u+1$ coefficient problems have extension-stable reductions where all $\mathbb{D}'$-critical problems are $\mathbb{D}'$-complete. Since $(\mathbb{D}(s_1)\dots(s_r),\sigma)$ is a Σ^*-extension of $(\mathbb{D},\sigma)$, $(\mathbb{B}(t)(s_1)\dots(s_r),\sigma)$ is a Σ^*-ext. of $(\mathbb{B}(t),\sigma)$ by Lemma 3.2. Moreover, $(\mathbb{S}'(t),\sigma)$ is a δ-optimal $\Pi\Sigma^*$-ext. of $(\mathbb{S}',\sigma)$: if t is a Π-ext., this follows by definition; otherwise, since $(\mathbb{B}(t),\sigma)$ is a δ-optimal ext. of $(\mathbb{B},\sigma)$, this follows by Lemma 4. Since reordering below of t does not change this property, $(\mathbb{B}'(t),\sigma)$ is a δ-optimal ordered $\Pi\Sigma^*$-ext. of

$(\mathbb{G},\sigma)$. Applying these arguments $b+1$-times shows that there is an extension in which all coefficient problems have extension-stable reductions and where all $\mathbb{D}'$-critical problems are $\mathbb{D}'$-complete. Since b and d are extension-stable, we obtain an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{B}'(t))$ to $\mathbb{D}'$ where all $\mathbb{D}'$-problems are $\mathbb{D}'$-complete. If $(\mathbb{G},\sigma)$ is δ-computable, Thm. 8 can be applied constructively. Hence such an extension can be computed; see Alg. 1. □

Algorithm 1. `CompleteSubProblems`$(\boldsymbol{a},\boldsymbol{f},\mathbb{E},\mathfrak{d})$

In: A δ-optimal ordered $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of a δ-computable $(\mathbb{G},\sigma)$ where $0\leq\mathfrak{d}<\delta(\mathbb{E})$ and $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ with $\delta(\mathbb{F})=\mathfrak{d}$, $\delta(t_i)>\mathfrak{d}$; a homogeneous $\boldsymbol{a}=(a_1,a_2)\in\mathbb{E}^2$, $\boldsymbol{f}\in\mathbb{E}^n$. An algorithm with the specification as Alg. 2.

Out: $(\mathbb{E}',B,S)$. A δ-optimal ordered $\Pi\Sigma^*$-extension $(\mathbb{E}',\sigma)$ of $(\mathbb{G},\sigma)$ with $\mathbb{E}':=\mathbb{D}(t_1)\dots(t_e)$ s.t. reordering of $(\mathbb{E}',\sigma)$ gives a Σ^*-extension of $(\mathbb{E},\sigma)$ with maximal depth $\mathfrak{d}$. A basis B of $\mathbb{V}$. The $\mathbb{D}$-critical problems S, all $\mathbb{D}$-complete, of an extension-stable reduction from $(\boldsymbol{a},\boldsymbol{f},\mathbb{D}(t_1)\dots(t_e))$ to $\mathbb{D}$.

(1)IF $e=0$ RETURN $(\mathbb{E}',B,\{(\boldsymbol{a},\boldsymbol{f})\})$ after computing $(\mathbb{E}',B)$:=`CompleteSolutionSpace`$(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$. FI

(2)Write $\mathbb{H}:=\mathbb{F}(t_1)\dots(t_{e-1})$. Compute an extension-stable denominator bound $d\in\mathbb{H}[t_e]^*$ of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t_e))$. Set $\boldsymbol{a'}:=(a_1/\sigma(d),a_2/d)\in\mathbb{H}(t_e)^2$, $\boldsymbol{f'}:=\boldsymbol{f}$ and clear the denominators and common factors. Compute an extension-stable degree bound b of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{H}[t_e])$.

(3)FOR $\delta:=b$ to 0 DO

(4) Define $\boldsymbol{0}\neq\boldsymbol{\tilde{a}_\delta}\in\mathbb{H}^2$ and $\boldsymbol{\tilde{f}_\delta}\in\mathbb{H}^n$ as in (5).

(5) IF $\boldsymbol{\tilde{a}_\delta}$ is inhomogeneous over $\mathbb{H}$ THEN compute a basis B_δ of $\mathrm{V}(\boldsymbol{\tilde{a}_\delta},\boldsymbol{\tilde{f}_\delta},\mathbb{H})$ and set $S_\delta=\{\}$, ELSE $(\mathbb{H},B_\delta,S_\delta):=$ `CompleteSubProblems`$(\boldsymbol{\tilde{a}_\delta},\boldsymbol{\tilde{f}_\delta},\mathbb{H},\mathfrak{d})$. FI

(6) Take the δ-optimal $\Pi\Sigma^*$-extension $(\mathbb{H}(t_e),\sigma)$ of $(\mathbb{H},\sigma)$ and define $\boldsymbol{f_{\delta-1}}$ by (6) or $\boldsymbol{f_{\delta-1}}:=(0)$.

(7)OD

(8)Compute a basis B_{-1} of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f_{-1}},\{0\})$ (base case II). Given the bases B_i, compute for $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}[t_e]_b)$ a basis $B=\{(\kappa_{i1},\dots,\kappa_{in},p_i)\}_{1\leq i\leq\mu}$; set $S:=\bigcup_{i=0}^b S_i$.

(9)RETURN $(\mathbb{H}(t_e),S,\{(\kappa_{i1},\dots,\kappa_{in},\frac{p_i}{d})\}_{1\leq i\leq\mu})$. FI

Example 5. We apply our algorithm for $\boldsymbol{a}=(1,-1)$, $\boldsymbol{f}=(\sigma(t_3/t_1^3))$ with the δ-optimal Σ^*-extension $(\mathbb{Q}(t_1)(t_2)(t_3),\sigma)$ of $(\mathbb{Q},\sigma)$ given in Exp. 1.1. Denote $\mathbb{D}=\mathbb{Q}(t_1)(t_2)$. We compute the denominator bound 1, the degree bound 2, and the first $\mathbb{D}$-critical problem $P_2:=((1,-1),(0))$; it is $\mathbb{D}$-complete. Hence, $\mathbb{D}$ is not extended. Next, we get the $\mathbb{D}$-critical problem $P_1:=(\boldsymbol{a'},(-2(1+(t_1+1)t_2),1))$ with $\boldsymbol{a'}=((t_1+1)^3,-(t_1+1)^3)$. We compute the δ-optimal Σ^*-extension $(\mathbb{D}',\sigma)$ of $(\mathbb{Q},\sigma)$ with $\mathbb{D}':=\mathbb{Q}(t_1)(t_2)(x_1)$ and $\sigma(x_1)=x_1+\frac{1}{(t_1+1)^3}$ s.t. P_1 is $\mathbb{D}'$-complete; see Exp 6.1. By Lemma 4 we can take the δ-optimal extension $(\mathbb{D}'(t_3),\sigma)$ of $(\mathbb{D}',\sigma)$ and get $P_0:=(\boldsymbol{a'},(1+(t_1+1)t_2,-x_1(1+(t_1+1)t_2),1))$ as the last $\mathbb{D}'$-critical problem; like in Exp. 4 one can test that P_0 is $\mathbb{D}$-complete. Hence, we get an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{D}'(t_3))$ to $\mathbb{D}'$ with the $\mathbb{D}'$-complete problems P_i. A basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{D}'(t_3))$ is $\{(0,1)\}$.

Lemma 8. *Suppose that Thm. 8 holds with the restriction that $\delta(\mathbb{F})=\mathfrak{d}-1$. Let $(\mathbb{E},\sigma)$ be a δ-optimal ordered $\Pi\Sigma^*$-extension of $(\mathbb{G},\sigma)$ where $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ with $\delta(\mathbb{F})=d-1$ and $\delta(t_i)=\mathfrak{d}$; let $\boldsymbol{a_1},\dots,\boldsymbol{a_n}\in\mathbb{F}^2$ be homogeneous and let $f_1,\dots,f_n\in\mathbb{F}$. Then there is a Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{F},\sigma)$ with maximal depth $\mathfrak{d}$ which can be reordered to a δ-optimal ordered Σ^*-extension of $(\mathbb{G},\sigma)$ with the following property: there are $g_i\in\mathbb{S}$ with $\sigma_{\boldsymbol{a_i}}g_i=f_i$ for all i. If $(\mathbb{G},\sigma)$ is δ-computable, such an extension can be computed.*

Proof. Suppose that the existence of such an extension $(\mathbb{S},\sigma)$ of $(\mathbb{F},\sigma)$ is proven for the first $n\geq 0$ cases. Take an additional homogeneous $\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}^2$ and $f\in\mathbb{F}$. If there is a $g\in\mathbb{S}$ with $\sigma_{\boldsymbol{a}}g=f$, we are done. Otherwise, by reordering of $(\mathbb{S},\sigma)$ we get a δ-optimal ordered $\Pi\Sigma^*$-ext. $(\mathbb{D}(x_1)\dots(x_r),\sigma)$ of $(\mathbb{G},\sigma)$ with $\delta(\mathbb{D})=\mathfrak{d}-1$ and $\delta(x_i)=\mathfrak{d}$. Take an $h\in\mathbb{F}^*$ with $\sigma_{\boldsymbol{a}}h=0$. By Lemma 7 we can take a Σ^*-ext. $(\mathbb{S}',\sigma)$ of $(\mathbb{S},\sigma)$ with maximal depth $\mathfrak{d}-1$ that can be reordered to a δ-optimal ordered $\Pi\Sigma^*$-ext. $(\mathbb{D}'(x_1)\dots(x_r),\sigma)$ of $(\mathbb{G},\sigma)$ with $\delta(\mathbb{D}')=\mathfrak{d}-1$ s.t. the following holds: there is an extension-stable reduction of $((1,-1),(-f/(ha_2)),\mathbb{D}'(x_1)\dots(x_r))$ to $\mathbb{D}'$ where all $\mathbb{D}'$-critical problems are $\mathbb{D}'$-complete. If $\sigma_{\boldsymbol{a}}g=f$ for some $g\in\mathbb{D}'(x_1)\dots(x_r)=\mathbb{S}'$, we are done. Otherwise, take the Σ^*-ext. $(\mathbb{D}'(x_1)\dots(x_r)(x),\sigma)$ of $(\mathbb{D}'(x_1)\dots(x_r),\sigma)$ with $\sigma(x)=x-f/(ha_2)$ and $\delta(x)\leq\mathfrak{d}$. Then $\sigma_{\boldsymbol{a}}(hx)=f$. By Prop. 2 the Σ^*-ext. $(\mathbb{D}'(x_1)\dots(x_r)(x),\sigma)$ of $(\mathbb{D}'(x_1)\dots(x_r),\sigma)$ is δ-optimal; by reordering one gets a Σ^*-ext. $(\mathbb{S}'(x),\sigma)$ of $(\mathbb{E},\sigma)$ with maximal depth $\mathfrak{d}$. Suppose that $(\mathbb{G},\sigma)$ is δ-computable. Then such g,h can be computed and Lemma 7 becomes constructive. Hence also Lemma 8 gets constructive. □

Proof of Theorem 8. The proof will be done by induction on $\delta(\mathbb{F})$. If $\mathfrak{d}=0$, $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{G}$-complete by Lemma 6.1. Now suppose that the theorem holds for $(\mathbb{F},\sigma)$ with $\delta(\mathbb{F})=\mathfrak{d}-1$, $\mathfrak{d}>0$. Consider the δ-optimal $\Pi\Sigma^*$-ext. $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ and $\delta(t_i)=\mathfrak{d}$; let $\boldsymbol{a}\in\mathbb{E}^2$ be homogeneous and $\boldsymbol{f}\in\mathbb{E}^n$. By Lemma 7 there is a Σ^*-ext. $(\mathbb{S},\sigma)$ of $(\mathbb{E},\sigma)$ with maximal depth $\mathfrak{d}-1$ which can be reordered to a δ-optimal ordered $\Pi\Sigma^*$-ext. $(\mathbb{D}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{G},\sigma)$ with the following property: there is an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{D}(t_1)\dots(t_e))$ to $\mathbb{D}$ s.t. all $\mathbb{D}$-critical problems, say $S=\{(\boldsymbol{a_i},\boldsymbol{f_i})\}_{1\leq i\leq k}$ with $\boldsymbol{f_i}=(f_{i1},\dots,f_{ir_i})\in\mathbb{D}^{r_i}$, are $\mathbb{D}$-complete. Lemma 8 shows that there is a Σ^*-ext. $(\mathbb{S}',\sigma)$ of $(\mathbb{E},\sigma)$ with maximal depth $\mathfrak{d}$ which can be reordered to a δ-optimal ordered $\Pi\Sigma^*$-ext. $(\mathbb{D}'(t_1)\dots(t_e),\sigma)$ of $(\mathbb{G},\sigma)$ s.t. $\sigma_{\boldsymbol{a_i}}=f_{ij}$ for all i,j. By Thm. 9 $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{D}'(t_1)\dots(t_e)$-complete and hence $\mathbb{S}'$-complete. If $(\mathbb{G},\sigma)$ is δ-computable, Lemmas 7 and 8 are constructive. This leads to Alg. 2. □

Algorithm 2. `CompleteSolutionSpace`$(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$

In: A δ-optimal ordered $\Pi\Sigma^*$-ext. $(\mathbb{F},\sigma)$ of a δ-computable $(\mathbb{G},\sigma)$ with $\mathfrak{d}:=\delta(\mathbb{F})$; $\boldsymbol{0}\neq\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}^2$ and $\boldsymbol{f}\in\mathbb{F}^n$. An algorithm with the specification as Alg. 1.

Out: $(\mathbb{D},B)$. A δ-optimal ordered $\Pi\Sigma^*$-extension $(\mathbb{D},\sigma)$ of $(\mathbb{G},\sigma)$ s.t. reordering gives a Σ^*-ext. of $(\mathbb{F},\sigma)$ with maximal depth $\mathfrak{d}$ and $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{D}$-complete; a basis B of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{D})$.

(1)IF $\mathfrak{d}=0$, compute a basis B of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$; RETURN $(\mathbb{F},B)$.

(2)$(\mathbb{E},B,S):=$ `CompleteSubProblems`$(\boldsymbol{a},\boldsymbol{f},\mathbb{F},\mathfrak{d}-1)$.

(3)Following Lemma 8, construct a δ-optimal ordered $\Pi\Sigma^*$-extension $(\mathbb{D},\sigma)$ of $(\mathbb{G},\sigma)$ s.t. reordering gives a Σ^*-ext. of $(\mathbb{E},\sigma)$ with maximal depth $\mathfrak{d}$ and s.t. there are $g\in\mathbb{D}^*$ with $\sigma_{\boldsymbol{a'}}g=f'_i$ for all $(\boldsymbol{a'},\boldsymbol{f'})\in S$ and all f'_i from $\boldsymbol{f'}$.

(4)IF $\mathbb{E}=\mathbb{D}$ RETURN $(\mathbb{E},B)$.

(5)Compute a basis B' of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{D})$; RETURN $(\mathbb{D},B')$.

Remark 1. If one always skips line (3) (during the recursion), one obtains the reduction presented in Section 4.

Example 6. Consider $(\mathbb{Q}(t_1)(t_2)(t_3),\sigma)$ from Exp. 1.1.
(1) We solve $\mathbf{C}$ for $(\mathbb{Q}(t_1)(t_2),\sigma)$, $\boldsymbol{a}=((t_1+1)^3,-(t_1+1)^3)$ and $\boldsymbol{f}=(-2(1+(t_1+1)t_2),1)$ by applying Algorithm 2:
(i) Alg. 1 computes for $(\boldsymbol{a},\boldsymbol{f},\mathbb{Q}(t_1)(t_2),1)$ the $\mathbb{Q}(t_1)$-critical problems $(\boldsymbol{a},\boldsymbol{f})$ with $\boldsymbol{f}\in\{(0,0),(-2(t_1+1)^2,0,-2(t_1+1)),((t_1+1)^2,1)\}$; they are all $\mathbb{Q}(t_1)$-compete by Lemma 6.2.
(ii) Next, the Σ^*-extension $(\mathbb{D},\sigma)$ of $(\mathbb{Q}(t_1)(t_2),\sigma)$ with maximal depth 2 is computed s.t. for any $g\in\mathbb{D}$ we have $(t_1+1)^3\sigma(g)-(t_1+1)^3g=f$ for $f\in\{0,-2(t_1+1)^2,-2(t_1+1),1\}$. We get $\mathbb{D}=\mathbb{Q}(t_1)(t_2)(x_1)(x_1')$ with $\sigma(x_1)=x_1+\frac{1}{(t_1+1)^3}$ and $\sigma(x_1')=x_1'+\frac{1}{(t_1+1)^2}$. By Thm. 9, $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{D}$-complete.
(iii) Finally, we compute for $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{Q}(t_1)(t_2)(x_1)(x_1'))$ the basis $\{(0,0,1),(0,1,x_1)\}$. By Thm. 5 $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{Q}(t_1)(t_2)(x_1)$-complete, i.e., we can remove the extension x_1'.
(2) We solve $\mathbf{C}$ for $(\mathbb{Q}(t_1)(t_2)(t_3),\sigma)$, $\boldsymbol{a}=(1,-1)$ and $\boldsymbol{f}=(\sigma(t_3/t_1^3))$ by applying Algorithm 2:
(i) We run Alg. 1, see Exp. 5, and get the δ-optimal ordered Σ^*-extension $(\mathbb{D}'(t_3),\sigma)$ of $(\mathbb{Q},\sigma)$ with the $\mathbb{D}'$-critical problems P_2,P_1,P_0 for the reduction $(\boldsymbol{a},\boldsymbol{f},\mathbb{D}'(t_3))$ to $\mathbb{D}'$. Note that for all $f\in\{0,1,-2(1+(t_1+1)t_2),x_1(1+(t_1+1)t_2)\}$ there is a $g\in\mathbb{D}'(t_3)$ with $(t_1+1)^3\sigma(g)-(t_1+1)^3g=f$, except the last entry, say f'.
(ii) We run Alg. 1 and obtain $(\mathbb{D}'(t_3),\{(0,1)\})$ for the input $(((t_1+1)^3,-(t_1+1)^3),(f'),\mathbb{D}'(t_3),2)$. Next we construct the Σ^*-ext. $(\mathbb{D}'(t_3)(x_2),\sigma)$ with $\sigma(x_2)=x_2+\frac{f'}{(t_1+1)^3}$. (It is δ-optimal by Prop 2; $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{D}'(t_3)(x_2)$-complete by Thm. 9.)
(iii) Finally, we get the solution in Exp. 1.1 by computing a basis for $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{D}'(t_3)(x_2))$. Summarizing, we have solved $\mathbf{B}$ for $(\sigma(t_3/t_1^3))$. In particular, we have represented the lhs of (2) in a δ-optimal Σ^*-extension.

Improvements of Alg. 2: (1) Skip (3) if there are $n+1$ elements in B; see Lemma 6.1. **(2)** Modify (2) if $\delta(\boldsymbol{f}),\delta(\boldsymbol{a})<\mathfrak{d}-1$: Write $\mathbb{F}=\mathbb{H}(t_1)\dots(t_e)$ where $\delta(t_i)=\mathfrak{d}$, $\delta(\mathbb{H})=\mathfrak{d}-1$, and set $(\mathbb{H}',B):=\texttt{CompleteSolutionSpace}(\boldsymbol{a},\boldsymbol{f},\mathbb{H})$. Afterwards, construct the $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{H}',\sigma)$ with $\mathbb{E}:=\mathbb{H}'(t_1)\dots(t_e)$ by Lemma 3.2. By Lemma 6.3 $(\boldsymbol{a},\boldsymbol{f})$ is $\mathbb{E}$-complete. **(3)** Similarly, speed up the computations in step (3) (Lemma 8): if $\delta(f_i),\delta(\boldsymbol{a'})<\mathfrak{d}-1$, do all the computations in $(\mathbb{H},\sigma)$. **(4)** Remove redundant extensions of $(\mathbb{D},\sigma)$ in step (5) by applying Thms. 4 and 5; see Exp. 6.1.

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