Local Parametrization of Cubic Surfaces ⋆

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Abstract

Algebraic surfaces – which are frequently used in geometric modeling – are represented either in implicit or parametric form. Several techniques for parameterizing a rational algebraic surface as a whole exist. However, in many applications, it suffices to parameterize a small portion of the surface. This motivates the analysis of local parametrizations, i.e., parametrizations of a small neighborhood of a given point P of the surface S. In this paper we introduce several techniques for generating such parameterizations for nonsingular cubic surfaces. For this class of surfaces, it is shown that the local parametrization problem can be solved for all points, and any such surface can be covered completely.

Key words: Parametrization; Cubics; Algorithm; Surfaces

1 Introduction

Many techniques from geometric modeling and Computer Aided Design are based on algebraic surfaces. Typically, these surfaces are described as the zero set of an algebraic equation (implicit representation), or as the image of map given by rational functions (parametric representation). Since both

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representations are appropriate for solving different types of problems, the automatic transition between these two representations is very important.

For instance, surface/surface–intersections can be traced efficiently if one of the surfaces is given in implicit form, and the other in parametric form. Another example is the detection of self–intersections of a surface, which becomes much simpler if both representations (implicit and parametric) are available. Algebraic methods for enhancing the performance of intersection algorithms in Computer–Aided Design are currently under investigation in a European project (Dokken et al., 2001).

Various techniques for generating a rational parametric representation of rational algebraic surfaces (called *parametrization* for short) are available, see Bajaj et al. (1998), Schicho (1998b) and Sederberg and Snively (1987). The reverse process is called *implicitization*. The implicitization problem is always solvable, and there are several different approaches to deal with this problem, as described by Busé (2001), Buchberger (1988), Corless et al. (2001), Dokken (2001) and Zheng et al. (2003).

This paper is devoted to general cubic surfaces, which have both an implicit and a rational parametric representation, except for the cone over an elliptic planar cubic curve. This property may make them particularly useful in a number of geometric modeling operations. On the other hand, these surfaces are sufficiently general, since any real–valued function on $\mathbb{R}^3$ can efficiently be approximated by a piecewise cubic function which is continuously differentiable, using three–dimensional Clough–Tocher elements, see Hoschek and Lasser (1993).

In most cases, the existing parametrization methods produce a birational map. Many methods use the 27 lines on a nonsingular cubic surface for parameterizing it (Berry and Patterson, 2001; Sederberg and Snively, 1987). It should be noted, however, that the computation of the lines is not a simple problem (Bajaj et al., 1998; Sederberg, 1990).

Several parametrization methods cannot be applied to surfaces with two real components. In such situations, one either uses two disjoint parametrizations or a two-to-one parametrization (Sederberg and Snively, 1987). Since the mentioned parametrization methods can be used only for certain classes of cubics, a thorough case analysis is needed.

Algebraic techniques often parameterize the algebraic surface as a whole. In many applications (such as geometric modeling and related areas), however, it suffices to have a parametrization defined in some open subset in the parameter space that covers the intersection of the surface with a certain region of interest. In contrast to the classical problem, we will refer to this as the problem of *local parametrization*: find a parametrization of a small neighborhood of a given point of the surface.

In this paper, we give a method for computing local parameterizations of non–singular cubic surfaces. The method works without analyzing the system of lines on the cubic surface. It produces rational maps defined in some neighborhood of the origin in the plane with the property that the image is an open subset of a given nonsingular cubic surface containing a given point P.

Our method is symbolic-numeric. We are manipulating symbolic objects such as polynomials, points, lines etc. On the other hand we assume that the coefficients of the polynomials and the coordinates of the point P are floating point numbers, and we compute the final result (parametrization) with floating point coefficients.

Clearly, we have to deal with various complications caused by the fact that small numerical errors are unavoidable, e.g., when we substitute the point P into the given equation then we do not get exactly zero. The reason for the decision to work with floating point numbers comes from the applications: they usually require a result in floating points, even if the result is not exact. See section 4.6 for more comments on this issue.

We use three local parametrization techniques for cubic surfaces, which are called the 2–curve technique, the repeater technique, and the reflection technique. The first two techniques can be traced back to Manin (1986) and Abhyanker and Bajaj (1987). They are based on the classical theory of rational curves on cubic surfaces. Such curves may be generated as the intersection of the surface with the tangent plane at a generic surface point.

We give a complete geometrical analysis of the introduced techniques for nonsingular cubic surfaces, and we show that each of the three algorithms computes a local parametrization for a given cubic surface S is a nonsingular cubic surface, and P is a surface point. The computed parametrization is improper. Clearly, properness cannot be expected, since the so–called F_5 surface has no proper parametrization (Schicho, 1998a). No computation of the lines on the surface is needed.

The remainder of the paper is organized as follows. Section 2 recalls some basic facts about cubic surfaces and introduces the local parametrization problem. Section 3 is devoted to a certain property of surface points, which we call the "t–property". The three algorithms for local parametrization are described in Section 4. We analyze each technique and we show that they provide a local parametrization around a given surface point. Finally, Section 5 concludes the paper.

2 Preliminaries

After recalling some properties of cubic surfaces, we introduce the notion of local parametrizations.

2.1 Cubic surfaces

Throughout this paper we work in the real projective space. We will consider a nonsingular cubic surface S. It is given by its implicit form F with floating point coefficients (see also the remarks in Section 4.6). A point of the surface will be called *generic*, if it does not belong to one of the lines lying completely on the surface.

Cubic surfaces are the zero set of a polynomial of degree 3. It is known since 1849, when Cayley and Salmon published their famous theorem, that there are 27 lines lying completely on a nonsingular cubic surface. One may conclude this theorem from the fact, that the number of lines on a nonsingular cubic surface is equal to the number of double tangent planes of an arbitrary tangent cone to the surface (Henderson, 1960).

Schläfli classified the cubic surfaces with respect to the number of real lines on them. The nonsingular cubic surfaces can be divided into 5 types F_1, F_2, $\ldots$, F_5 with respect to the number of real lines (27, 15, 7, 3 and 3, respectively) and real components (1, 1, 1, 1 and 2, respectively).

Later, Schläfli classified the cubic surfaces (singular and non–singular ones) into 23 species with respect to the nature of the singularities on the surfaces. A complete classification with 21 classes over $\mathbb{C}$ has been given by Bruce and Wall (1979).

For future reference we recall that each non–singular cubic surface has at least one real line, and that surfaces consist of one $(F_1, \ldots, F_4)$ or two (F_5) real components. One of the two components of the F_5 surface is convex in the following sense:

Definition 1 *A connected component of a surface is said to be convex, if there exists an auxiliary plane, such that for any tangent plane of the component, the component is fully contained in one of the two cells defined by the planes.*

The auxiliary plane acts as the plane at infinity.

Figure 1 shows a surface with one real component and all the real lines on it, and Figure 2 shows a cubic surface of type F_5 with two components (both

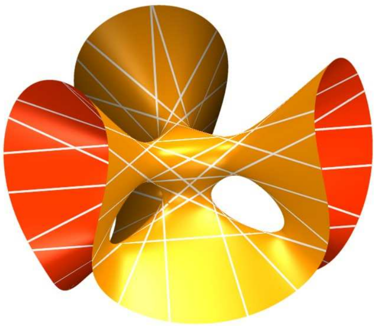

Figure 1. F_1 surface

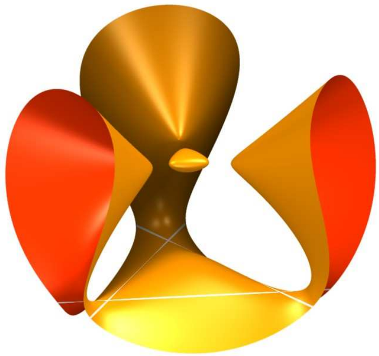

Figure 2. F_5 surface

pictures courtesy of O. Labs).

2.2 Local parametrization

Given the surface S and a point P on it, we are interested in finding a rational map defined in a certain neighborhood of the origin, which is 'well–behaved' at P, and covers a certain neighborhood of the given point.

Definition 2 *A quadruple of polynomials* $(\pi_1(u,v),\ldots,\pi_4(u,v))$ *is called a local parametrization of the surface* S *at the point* P*, if the image of the origin is* P*,*

$$(\pi_1(0,0) : \pi_2(0,0) : \pi_3(0,0) : \pi_4(0,0)) = (P_1 : P_2 : P_3 : P_4), \qquad (1)$$

and the image of the rational map defined by the four polynomials is fully contained the surface. The local parametrization is said to be regular, if the Jacobian matrix of the mapping

$$(u,v,\rho) \mapsto (\rho\,\pi_1(u,v),\ldots,\rho\,\pi_4(u,v)) \qquad (2)$$

has full rank (i.e., 3) at $(0:0:1)$.

The following result is an immediate consequence of the implicit mapping theorem, see e.g. Kendig (1977).

Proposition 3 *For any given regular local parametrization* G *there exists a neighborhood of the origin in the parameter space, such that the restriction of* G *to this neighborhood is faithful.*

3 Analyzing the system of all tangent planes

We analyze the location of points with respect to the system of all tangent planes of the given cubic surface.

3.1 The t-property

We introduce the following auxiliary notion.

Definition 4 *Let S be a cubic surface and $P \in S$ a generic point on the surface. P is said to have the t–property if P is contained in the tangent plane at another surface point.*

Clearly, this second tangent plane is different from the tangent plane at P.

We recall the definition of *contour generator* and *apparent contour* following Cipolla and Giblin (2000).

Using a perspective projection, we project a given surface $\bar{S}$ from a given point P into a plane Π, $P \notin \Pi$. The point P is called the *center of the projection*, while Π acts as the *image plane*.

We consider the cone of lines through P which are tangent to the surface $\bar{S}$. This cone is called the *tangent cone* to $\bar{S}$ with apex P. The curve on $\bar{S}$ where this cone is tangent to $\bar{S}$ is called the *contour generator*, and the curve where the cone intersects the image plane is the *apparent contour*.

In the case of a cubic surface S, the contour generator is a space curve of degree 6, and the apparent contour is a planar quartic curve. In fact, if we move the point P (i.e., the center of the projection) to the origin $(0:0:0:1)$, the equation of the surface takes the form

$$x_4^2 L(x_1, x_2, x_3) + x_4 Q(x_1, x_2, x_3) + K(x_1, x_2, x_3) = 0, \qquad (3)$$

where L, Q and K are linear, quadratic and cubic homogeneous polynomials, respectively. After a short computation one arrives at the the equation

$$[Q(x_1, x_2, x_3)]^2 - 4L(x_1, x_2, x_3)\, K(x_1, x_2, x_3) = 0 \qquad (4)$$

of the apparent contour.

First we analyze the singularities which may be present in the apparent contour.

Lemma 5 *The apparent contour associated with a point P on a non–singular cubic surface has a singular point if and only if the point P lies on one of the lines on the surface.*

Proof. Let $P = (0 : 0 : 0 : 1)$, and assume that equation of the surface has the form (3). We may assume that $L = x_3$, i.e., that the tangent plane at P is $x_3 = 0$.

First case: The apparent contour has a singular point in the tangent plane at P. Without loss of generality we assume that it is located at $(1 : 0 : 0)$. A short computation reveals that this implies $q_{2,0,0} = k_{3,0,0} = 0$, where q_{ijk} and k_{ijk} are the coefficients of $x_1^i x_2^j x_3^k$ in Q and K, respectively. Consequently, the line $(0 : 0 : s : t)$ $(s, t \in \mathbb{R})$ is fully contained in the surface.

Second case: The apparent contour has a singular point which is not in the tangent plane at P. Without loss of generality we assume that it is located at $(0 : 0 : 1)$. A short computation reveals that this implies $k_{0,1,2} = \frac{1}{2} q_{0,0,2} q_{0,1,1}$, $k_{1,0,2} = \frac{1}{2} q_{0,0,2} q_{1,0,1}$ and $k_{0,0,3} = \frac{1}{4} q_{0,0,2}^2$. The surface is singular, since it has the singular point $(0 : 0 : -2 : q_{0,0,2})$.

Finally, it can be shown that any line through P generates a singular point of the apparent contour. □

The t–property can now be characterized by using the apparent contour.

Proposition 6 *A generic point P of the cubic surface S has the t–property if and only if the apparent contour of the surface with center P has real points.*

Proof: The point P has the t–property, if and only if there exists a point $R \in S$, $R \neq P$, such that $P \in T_R S$, where $T_R S$ is the tangent plane to the surface S at R. This is equivalent to the fact that the line connecting P and R is a real line of the tangent cone with apex P. This line corresponds to a regular point of the apparent contour. Note that the apparent contour cannot have singularities, since P is assumed to be a generic point (cf. Lemma 5). □

The following algorithm, which is based on Proposition 6, is needed for computing the local parametrizations, as described in the next sections.

Algorithm 1 `(point on contour)`
Given: An implicit equation F of a cubic surface S and a point $P \in S$.
Synopsis: Decide t–property for P. If P has t–property find $R \in S$, such that $P \in T_R S$.

(1) We move P to the origin by an orthogonal transformation of the homogeneous coordinates, and compute the equation of the apparent contour (4).
(2) Check whether the apparent contour has non–singular real points using

methods similar to Gonzalez-Vega and Necula (2002).

(a) If the apparent contour does not have non–singular real points, then the algorithm stops. The point P does not have the t–property.

(b) If the apparent contour has non–singular real points, then P has t–property. Go to the next step.

(3) Find a nonsingular real point $(\bar{x}_1 : \bar{x}_2 : \bar{x}_3)$ on the apparent contour (cf. Gonzalez-Vega and Necula, 2002).

(4) Compute the corresponding point R on the contour curve. If P is at $(0 : 0 : 0 : 1)$, then

$$\bar{R} = (\bar{x}_1 : \bar{x}_2 : \bar{x}_3 : \frac{-Q(\bar{x}_1, \bar{x}_2, \bar{x}_3)}{2L(\bar{x}_1, \bar{x}_2, \bar{x}_3)}). \tag{5}$$

(5) Using the inverse of the orthogonal transformation in (1), we transform $\bar{R}$ to get $R \in S$.

(step 1) Here we use an orthogonal transformation in order to keep numerical errors small. The cubic term x_4^3 appearing in the transformed equation of the surface is due to numerical errors and it is set to zero.

(step 4) The point $\bar{R}$ is both on the line connecting P and $(\bar{x}_1 : \bar{x}_2 : \bar{x}_3 : 0)$ and on the surface.

3.2 Locating the points with t–property

Given a non–singular cubic surface, we identify the regions of points with and without t–property on a nonsingular cubic surface.

Lemma 7 *The regions on a cubic surface S containing points with and without t–property are bounded by the real lines of S.*

Proof: Consider the system apparent contours associated with all points on the surface. Clearly, the coefficients of these planar curves depend continuously on the location of the points.

If one moves along a curve from a point P with t-property to a point Q without it, the apparent contour, which has at least one real component at P, has first to degenerate to a singular point, before disappearing eventually. Due to Lemma 5, this takes place exactly when one crosses one of the lines lying on the surface. □

Depending on the local behavior of a surface with respect to its tangent plane at a point, one arrives at different types of surface points. We assume that we have a non–flat surface point, i.e., $Q \not\equiv 0$ in (3). A point is called *elliptic* if the tangent plane at the point intersects the surface in an isolated point, *hyperbolic* if the tangent plane intersects the surface (locally) in a pair of intersecting curves with two different tangents, and *parabolic* otherwise.

One may distinguish the three types of points by the Gaussian curvature K. A point of a surface is called *elliptic* if $K > 0$, *parabolic* if $K = 0$ and *hyperbolic* if $K < 0$.

The non–convex component of a cubic surface may consist of hyperbolic, elliptic, and parabolic points, while the convex component of the F_5 surface has elliptic points only.

Lemma 8 *Generic hyperbolic points of a cubic surface have the t–property.*

Proof: A short computation reveals that the two asymptotic directions at generic hyperbolic point (i.e., the tangent directions of the two branches of the intersection curve with the tangent plane at the point) correspond to real points of the apparent contour. □

Theorem 9 *A point of a nonsingular cubic surface S has the t–property if and only if it lies on the non-convex component of the surface.*

Proof: Any non-singular cubic surface contains at least 3 real lines. Lines are always on the non-convex component of S, as the convex component does not contain any line. These lines define a partition of the component into several cells.

Any line of a nonsingular cubic surface contains only hyperbolic points, with the exception of the two parabolic points (Segre (1942)). Consequently, the neighborhood of any line contains hyperbolic points which have t–property (Lemma 8). According to Lemma 7, if a point has t–property, then this property is shared by all points in the cell.

It remains to be shown that the convex component of the F_5 surface does not contain points with t–property. Consider any point $R \in S$ on the convex component. Assume, that there is a point $Q \in S$ such that $R \in T_Q S$. As the tangent plane cannot intersect the convex component in a different point than Q, the point Q cannot lie on the convex component. The line in $T_Q S$ connecting Q and R has four intersections with the surface S, since Q has to be counted twice. This is a contradiction, since any line has at most three real intersections with a cubic surface. □

4 Three techniques for generating local parametrization

We describe three approaches to the solution of the local parametrization problem. The three techniques are based on the theory of rational curves on cubic surfaces. For the convenience of the reader, we summarize it in the next section.

4.1 Rational cubics on cubic surfaces

The intersection of a cubic surface with the tangent plane at a generic surface point P always gives a rational planar cubic, where the point will be the singular point of the curve. A rational cubic can be parameterized by a pencil of lines through the singularity of the curve, which intersect the cubic at exactly one other point. The coordinates of the latter point give parametric functions for the cubic curve.

More precisely, if we assume that $P = (0 : 0 : 0 : 1)$ and that the tangent plane at the origin equals $x_1 = 0$, the equation of the surface takes the form

$$x_4^2 x_1 + x_4 Q(x_1, x_2, x_3) + K(x_1, x_2, x_3).$$

The cubic curve C_P cut by the tangent plane at the origin is

$$Q(0, x_2, x_3)x_4 + K(0, x_2, x_3).$$

It has the rational parametrization $(0 : t : 1 : -K(0, t, 1)/Q(0, t, 1))$. See Abhyankar and Bajaj (1988) for further details.

4.2 The 2–curve technique

This technique has been described by Manin (1986). Let Q_1 and Q_2 be two real points on the cubic surface S as in Figure 3. We denote by C_{Q_i} the curves cut by the tangent plane $T_{Q_i}S$, $i = 1, 2$, from the surface S. The cubic curves C_{Q_i} have a double point at Q_i, therefore they can be parameterized by rational functions.

Let $\pi_i : \mathbb{R} \to C_{Q_i}$ be the parametrization of the i–th curve. Then $\pi : \mathbb{R}^2 \to S$, $(s, t) \mapsto R$ gives a parametrization of a neighborhood of R, where R is the third point of the surface obtained by intersecting with the line $\pi_1(s), \pi_2(t)$.

This idea is formalized in the following algorithm.

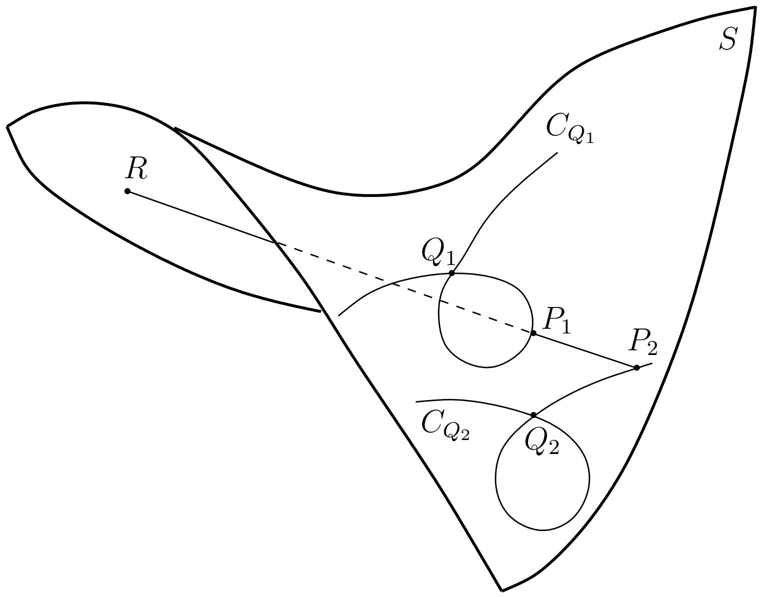

Figure 3. The 2–curve technique

Algorithm 2 `(2-curve technique)`
Given: An implicit equation F of a cubic surface S and a point $P \in S$.
Synopsis: Find 4 polynomials depending on 2 parameters, which define a local parametrization of S around P.

(1) Check t–property for P
   (a) If P has no t–property the algorithm stops.
   (b) If P has t–property go to the next step.
(2) Choose random lines through P until one of the lines generates 2 further intersection points P_1, P_2 with t-property.
(3) (a) Choose randomly a point Q_1 on the contour of P_1. (see Algorithm 1)
   (b) Choose randomly a point Q_2 on the contour of P_2 such that the tangents at P_1 and P_2 to the curves C_{Q_1} and C_{Q_2} are not coplanar.
(4) Parameterize the cubics $C_{Q_i} = S \cap T_{Q_i}S$, such that the parameter 0 corresponds to P_i, $i = 1, 2$.
(5) Let the parametrization of C_{Q_i} be $(x_i(t_i) : y_i(t_i) : z_i(t_i) : w_i(t_i))$. Intersect the line $(x_1(t_1) + \lambda x_2(t_2) : y_1(t_1) + \lambda y_2(t_2) : z_1(t_1) + \lambda z_2(t_2) : w_1(t_1) + \lambda w_2(t_2))$ with S; this leads to a quadratic equation with one root at 0. Compute the remaining root $\lambda(t_1, t_2)$ and substitute it back into the equation of the line. This gives the parametrization of the surface S around the point P.

We give a more detailed description for some steps of Algorithm 2.

(step 2) Let denote by $(p_1 : \cdots : p_4)$ the coordinates of P, and let S given by F. Then a random line l_r^P through P is given by $(p_1 + \mu \cdot r_1 : \cdots : p_4 + \mu \cdot r_4)$, where $r_1, \ldots, r_4$ are randomly generated real numbers. To compute the intersection of S and l_r^P we substitute the equation of the line into the equation of the surface $F(l_r^P)$, and compute the solutions for μ.

Since one of the solutions is equal to zero, we divide by μ and obtain a quadratic equation. Solving it numerically we get the further intersection points of S and l_r^P.

(step 5) As the last step of the algorithm, we have to intersect the line $C_{1,x_i} + \lambda C_{2,x_i}$ with S, and compute the values of λ. After substituting the equation of the line into F and eliminating the cubic and constant terms with respect to λ, which are caused by numerical errors, we get $B_1(t_1, t_2)\lambda + B_2(t_1, t_2)\lambda^2$. Hence, $\lambda = -B_1/B_2$.

We apply the algorithm to an example.

Example 1 Consider the surface S defined by

$$F = 3x_4x_1^2 + 3x_4x_2^2 + 3x_4x_3^2 - 10x_1x_2x_3 - 3x_4^3,$$

and a point $P = (1 : 3 : 27 : 27)$ on S. Using Algorithm 1 we check that the point P has t–property. We take a random line through P and intersect it with the surface S, giving 2 additional points on S:

$$P_1(-5.98687 : -3.36937 : -1.87809 : -5.09084),$$
$$P_2(-1.07744 : -0.44309 : 1.44983 : 0.83569)$$

We want to compute two points Q_1, Q_2, such that P_i is on the tangent plane $T_{Q_i}S$. For this we have to compute a point on the contour curve of S with respect to the projection from P_i.

The apparent contour of S with respect to the projection from P_1 is:

$$K_1 = (-15.27251x_1^2 - 15.27251x_2^2 - 15.27251x_3^2 + 18.78088x_1x_2 + 33.69367x_1x_3 + 59.86867x_2x_3)^2 + 40(119.58934x_1 - 9.52124x_2 - 144.35332x_3)x_1x_2x_3.$$

$Q_1^p(6.53995 : 5.94157 : -2.97076 : 1)$ is a point on K_1, which corresponds to the point

$$Q_1(0.55308 : 2.57216 : -4.84885 : -5.09084)$$

on S. The intersection of S with the tangent plane at Q_1 gives a curve C_1. The parametrization of C_1 is:

$$\begin{pmatrix} 2.15038t_1^3 - 44.54884t_1^2 + 216.56989t_1 - 263.60396 \\ 4.48066t_1^3 - 74.41867t_1^2 + 205.08880t_1 - 148.34822 \\ -18.79020t_1^2 + 75.48726t_1 - 82.69451 \\ -24.43222t_1^2 + 160.93907t_1 - 224.14926 \end{pmatrix}$$

The apparent contour of S with respect to the projection from P_2 is:

$$K_2 = (1.25354x_1^2 + 1.25354x_2^2 + 1.25354x_3^2 - 7.24916x_1x_2 + 2.21548x_1x_3 + 5.38720x_2x_3)^2 + 40(0.25543x_1 + 3.34983x_2 + 0.62389x_3)x_1x_2x_3.$$

Similarly, $Q_2^p(0.77339 : 0.19981 : -0.39962 : 1)$ is a point on K_2, which corresponds to the point

$$Q_2(0.23467 : -0.021736 : 0.32529 : 0.41785)$$

on S. The parametrization of C_2 is

$$\begin{pmatrix} -1.05829t_2^3 + 3.27466t_2^2 + 0.80973t_2 - 5.21309 \\ -0.85279t_2^3 - 0.88123t_2^2 + 4.94149t_2 - 2.14387 \\ 3.18407t_2^2 - 9.35177t_2 + 7.01474 \\ 5.18538t_2^2 - 10.67940t_2 + 4.04335 \end{pmatrix}$$

Let the coordinates of the curve C_1 be $(C_{1,x_1}, C_{1,x_2}, C_{1,x_3}, C_{1,x_4})$ and the coordinates of C_2 be $(C_{2,x_1}, C_{2,x_2}, C_{2,x_3}, C_{2,x_4})$. Let the equation of the line connecting $C_1(t_1)$ and $C_2(t_2)$ be $C_{1,x_i} + \lambda C_{2,x_i}$. Substituting this equation into F we get $B_1(t_1,t_2)\lambda + B_2(t_1,t_2)\lambda^2$. Thus $\lambda = -B_1/B_2$. Substituting it back into $C_{1,x_i}(t_1) + \lambda(t_1,t_2)C_{2,x_i}(t_2)$ gives a parametrization of a neighborhood of the point P.

Figure 4 shows several parameterized patches on a given cubic surface.

Now we prove the correctness of Algorithm 2.

Theorem 10 *For a nonsingular cubic surface S and generic point $P \in S$, Algorithm 2 produces a regular local parametrization if and only if P is a point with t–property.*

Proof: If P has t–property, then the whole component containing P has only points with this property. Thus we can always find lines through P which intersect the surface S in 2 additional real intersections with the non-convex component of the surface, i.e., in points with t–property.

Due to the construction of the algorithm, the image of the origin is P and the image of the map is contained in the surface.

It remains to be shown that – in step 3 – it is always possible to choose a point Q_2 on the contour curve such that the tangent lines to the curves C_{Q_1} and C_{Q_2} are not coplanar.

Let l_{P_1} denote the tangent line at P_1 to the curve C_{Q_1}. (l_{P_1} is the intersection of the tangent planes $T_{Q_1}S$ and $T_{P_1}S$.) Furthermore denote by $l_{P_2}^j$ the tangent

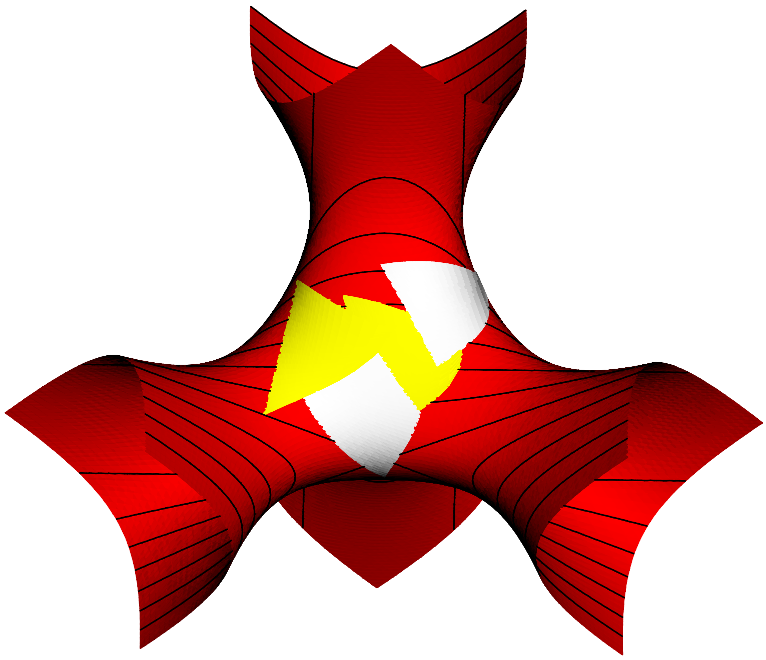

Figure 4. Implicit surface with parameterized patches

line at P_2 to the curve $C_{Q_2^j}$. (It is the intersection of the tangent planes $T_{Q_2^j}S$ and $T_{P_2}S$.) The line connecting P_2 with the intersection of l_{P_1} and $T_{P_2}S$ gives the tangent direction at P_2 which is forbidden. We have to show, that it is always possible to choose a point Q_2^j on the contour with respect to P_2 such that $l_{P_2}^j$ is not the forbidden direction.

We show that it is not possible to have the same tangent direction for all points on the apparent contour with respect to P_2. For each point on the apparent contour we get a corresponding point on the contour generator Q_2^j and a tangent direction $l_{P_2}^j$. If all points gave the same tangent direction l_{P_2}, then all tangent planes $T_{Q_2^j}S$ would go through this line, i.e. we would get a pencil of planes. Thus the envelope surface of these tangent planes would degenerate into a line, which is not possible.

As one can verify by direct computation, if the tangents to the curves C_{Q_1} and C_{Q_2} are not coplanar, then the Jacobian of the parametrization has full rank at P.

On the other hand, if P does not have the t–property, then Algorithm 2 stops in step 1. In this situation it is clear, then any line through P would intersect S in another point which does not have t–property. $\square$

Remark 1 It can be shown that the parametrization computed by Algorithm 2 has bidegree $(6, 6)$ and total degree 12.

We call a number $k \in \mathbb{N}$ the *index* of the parametrization if all points outside

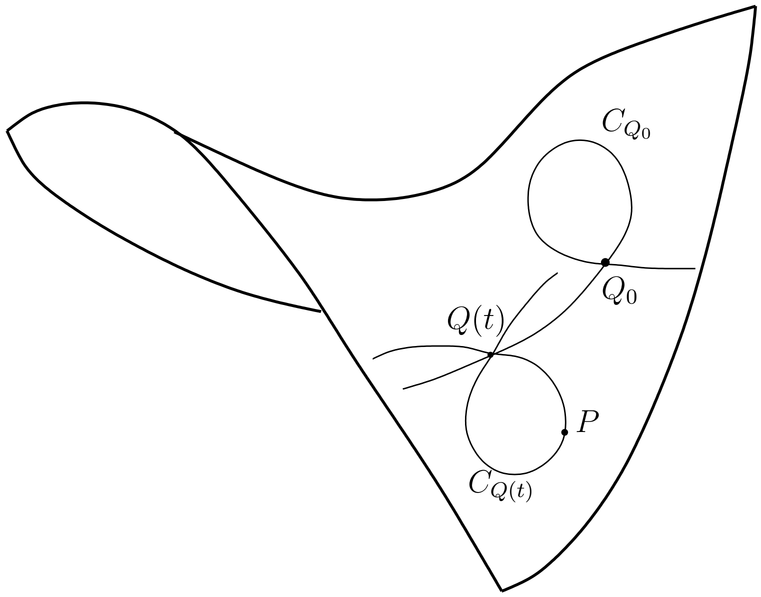

Figure 5. The repeater technique

a Zariski closed subset are generated by k complex parameter pairs (Sendra and Winkler (2001)). A proper parametrization has index 1.

Proposition 11 *The index of the parametrization obtained by the 2-curve technique equals 6.*

Idea of the proof: If P is a point on S that can be parameterized using the points Q_1, Q_2, we have to compute how many lines through P exist, which intersects both curves C_{Q_1}, C_{Q_2}. As the planes of the two curves C_{Q_1}, C_{Q_2} intersect in a line, the curves have three intersections on a line. If we project the two curves C_{Q_1}, C_{Q_2} from P on an arbitrary plane we get nine intersection points from which three are on a line (Bezout's theorem). Hence we can reach P six times using this parametrization method. See Manin (1986) for a complete proof. □

Remark 2 Methods for reducing the index of the parametrization of curves exist Sederberg (1984, 1986). Unfortunately, currently no methods for reducing the index in the surface case are available.

4.3 The repeater technique

Let Q_0 be a real point on S as in Figure 5. The rational cubic $C_{Q_0} = T_{Q_0}S \cap S$ has a rational parameterization $\pi_{Q_0} : \mathbb{R} \to C_{Q_0}$. Let $C_{Q(t)}$ be a curve cut by the tangent plane at the point $Q(t) := \pi_{Q_0}(t)$. Then, the parametrization of the curve $C_{Q(t)}$, $\pi_{Q(t)} : \mathbb{R} \to C_{Q(t)}, s \mapsto \pi_{Q(t)}(s)$ gives a parametrization of a neighborhood of the point $R := \pi_{Q(t)}(s)$.

The above technique leads to the following algorithm.

Algorithm 3 `(repeater technique)`
Given: An implicit equation F of a cubic surface S and a point $P \in S$.
Synopsis: Find 4 polynomials depending on 2 parameters, which give a local parametrization of S around P.

(1) Check t–property for P
 (a) If P has no t–property the algorithm stops.
 (b) If P has t–property go to the next step.
(2) Compute the contour generator K with respect to P and choose a point Q on it with t–property. (see Algorithm 1)
 (a) If there is no such point the algorithm stops.
 (b) If there is such a point go to the next step.
(3) Compute the contour generator with respect to Q and choose a point Q_0 on it such that $T_{Q_0}S$ does not contain the tangent at Q to K.
(4) Compute the intersection of S with tangent plane $T_{Q_0}S$: C_{Q_0}.
(5) Parameterize C_{Q_0}, such that $Q = \pi_{Q_0}(0)$.
(6) Parameterize the curve $C_{Q(t)}$, which is the intersection of S with the tangent plane at the point $Q(t) := \pi_{Q_0}(t)$, such that $P = \pi_{Q(0)}(0)$.

Remark 3 It may happen that the contour generator with respect to the point P is lies on a convex component of S. In this case the algorithm stops at step 2. In other words the t–property for P is not sufficient for Algorithm 3 to produce a result.

Remark 4 For parameterizing the curve $C_{Q(t)}$ in step 6, we do not use an orthogonal transformation as in step 5. The reason is that equation of the tangent plane $T_{Q(t)}S$ depends on the parameter t, and an orthogonal transformation moving it to the plane $x_1 = 0$ requires square roots. It is possible to compute a non–orthogonal transformation.

Remark 5 It is not difficult to prove that Algorithm 3 produces always a local parametrization, if the contour generator with respect to P has points with t–property. In order to obtain a *regular* local parametrization we need an additional condition analogous to the condition in Algorithm 2 that the tangents at P_1, P_2 to C_{Q_1} and C_{Q_2} are not coplanar. More precisely, the tangent plane at $T_{Q_0}S$ must not contain the tangent at Q to the contour generator with respect to P. We do not give a detailed proof, as the repeater technique is less useful than the 2–curve technique for our purposes.

Remark 6 The total degree of the parametrization using Algorithm 3 is 12. The index of the parametrization obtained by the repeater technique is 6, see Manin (1986).

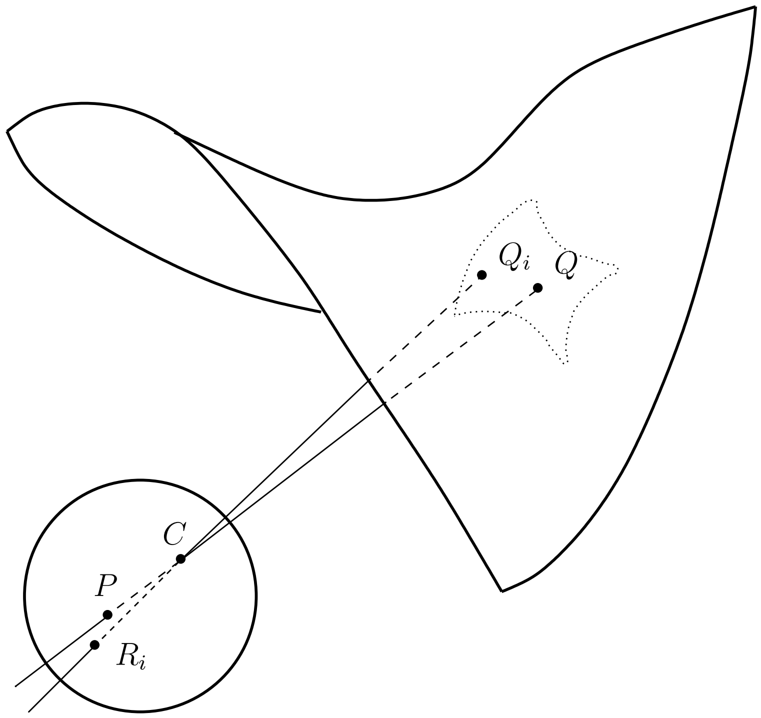

Figure 6. The reflection technique

4.4 The reflection technique

Algorithm 2 and Algorithm 3 fail if the surface has two components, and the parameterizable point is on the convex piece. We can detect this case simply by checking if the point has t–property or not. If it does not have this property, then it is located on the convex component. In such situations, we can use the following technique.

Let P be the point on the surface S; see Figure 6. Using Algorithm 2 we can parameterize some region of S. Connect the point P with any point from the parameterized region and denote by C the further intersection point with S. From C reflect the points of the parameterized region. This gives a parametrization of the neighborhood of the point P.

We summarize this idea in

Algorithm 4 `(reflection technique)`
Given: An implicit equation F of a cubic surface S and a point $P \in S$.
Synopsis: Find 4 polynomials depending on 2 parameters, which define a parametrization of a neighborhood of P.

(1) Using Algorithm 2 compute a local parametrization for a point $R \in S$ with t–property, where $R \notin T_P S$. Let the parametrization be $P_{t_1,t_2} := (X(t_1,t_2) : Y(t_1,t_2) : Z(t_1,t_2) : W(t_1,t_2))$

(2) Connect P with the point R, and intersect this line with S. Let $C(c_1 : c_2 : c_3 : c_4)$ be the further intersection point.
(3) Intersect the line CP_{t_1,t_2} with S. Compute the third point of intersection in terms of t_1, t_2.

Remark 7 In step 3 in Algorithm 4 we need to compute the third point of intersection of S with a line through two points of S. This can be done in the same way as in step 5 of Algorithm 2.

Example 2 We use the same surface as in Example 1. We want to construct a local parametrization around $P = (1.53295 : 53.20912 : 10.85109 : 1)$ Using Algorithm 2 we compute a local parametrization around the point $R = (1 : 3 : 27 : 27)$ as in Example 1. Let denote the computed parametrization by $(X(t_1,t_2) : Y(t_1,t_2) : Z(t_1,t_2) : W(t_1,t_2))$.
The parametric equation of the line connecting P and R is

$$(u + 1.53295 : 3u + 53.20912 : 27u + 10.85109 : 27u + 1).$$

Substituting it into the equation of S, and solving the resulting equation for u we get the further intersection of the line with S:

$$C(-2.46836 : 41.20518 : -97.18435 : -107.03544).$$

The line CP_{t_1,t_2} connecting C with the points of the parameterized region has the form

$$(\lambda X - 2.46836 :, \lambda Y + 41.20518 : \lambda Z - 97.18435 : \lambda W - 107.03544).$$

Substituting it into F we get an equation of the form $B_1(t_1,t_2)\lambda + B_2(t_1,t_2)\lambda^2$. Computing λ and substituting it back into the equation of CP_{t_1,t_2}, we get a local parametrization around P.

Theorem 12 *For a nonsingular cubic surface S and point $P \in S$, Algorithm 4 always gives a regular local parametrization.*

Proof: We have to show that it is always possible to find $R \in S$ with t–property, where $R \notin T_P S$. The intersection of S and $T_P S$ is a degree 3 curve. The non-convex component of S contains other points, and these points have the t–property. Let R be one of these points. As it was shown before, Algorithm 2 produces a regular local parametrization around the point R.

As $R \notin T_P S$, the line connecting P and R is not tangent at P. Let C be the third point of intersection of S with the line PR. Then the line PR intersects the two tangent planes at P and R transversally. This implies that the reflection of the cubic surface S at C restricts to a local isomorphism of sufficiently small regions of S around P and R. Therefore the composition of this map with the regular local parametrization around R is regular. □

The reflection technique works for any type of cubic surfaces. It can be applied arbitrarily, but it is particularly interesting in the situation, when the given surface has two components, and the parameterizable point lies on the convex part.

Remark 8 As one may verify by a straightforward computation, the parametrization computed by Algorithm 4 has bidegree $(12, 12)$ and total degree 24.

Proposition 13 *The index of the parametrization obtained by the reflection technique is 6.*

Proof: Reflection at a point is birational and does not change the index. As the index of the 2–curve technique is 6, the index of the reflection technique is also 6. □

4.5 Covering a surface by local parameterizations

Theorem 14 *Given a nonsingular cubic surface. It can be covered by finite number of local parameterizations.*

Proof: For each point P on the surface we can compute a local parametrization P_P, which covers some open neighborhood U_P of P. Obviously $S = \cup_{P \in S} U_P$. Since S is compact there exist a finite subcover. □

4.6 Exact computation

Can we extend the algorithms in the previous sections to the exact case? This means we assume that we have given the coefficients of the equation of S and the coordinates of the point P as exact rational numbers, and we want to compute a local parametrization which has exact coefficients.

In principle this is possible, but there are several obstructions. In Algorithm 1 we need to construct a point of a plane curve of degree 4. It can not be expected that there exist a rational solution, even if there exist it might be very hard to find it. However we can always construct an exact solution in a field extension of degree 4.

In step 2 of Algorithm 2 we need to solve a univariate equation of degree 2. This also introduces in general a field extension of degree 2.

In the algorithms we assume that we have a point $P \in S$ given. Numerically it is not a problem to compute such a point. However in the exact situation it is not trivial, see Manin (1986)

On the other hand if we compute with floating point numbers errors are unavoidable and should be considered, e.g. using tools as interval arithmetic (Moore, 1966), exact real arithmetic in the sense of (Gianantonio, 1993), or probabilistic arithmetic. A detailed error analysis is beyond of the scope of the present paper.

5 Conclusion

Using the techniques described in this paper, the vicinity of a generic point on a given surface can be covered by a regular rational parametrization. As a potential advantage, this parametrization is found without analyzing the type of the cubic surface, i.e., without discussing the system of the 27 lines.

It can be expected that the results extend non–generic regular points and to singular surfaces. However, the complete analysis of the singular cases is beyond the scope of this paper, since it requires the study of each surface class separately (20 cases over $\mathbb{C}$, and many more cases over $\mathbb{R}$). Further results will be presented in Szilágyi (2005)

As a matter of future research, the numerical stability of the method should be further explored. As observed in our experiments, the quality of the resulting parametrization can greatly be enhanced by using some heuristic ideas for optimizing the position of the randomly chosen lines, etc. In addition, whenever a special coordinate system has to be chosen, we use an orthogonal transformation of the homogeneous coordinates, in order to minimize the effect of rounding errors.

Another challenging problem is the use of exact symbolic computation throughout the algorithm. While the current implementation had to resort to floating point numbers, the underlying concepts are purely symbolic and would therefore benefit from a symbolic–computation–based implementation.

Finally, since the method produces improper parametrizations of a relatively high index, systematic techniques for reducing the index of a parametrization could be of some interest and should be explored.

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