An Approach to Online Parameter Estimation in Nonlinear Dynamical Systems

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Abstract

Given a dynamical system $\dot{y} = f(q_*, y, t)$ in a (possibly infinite dimensional) Hilbert space Y, we consider the problem of estimating the unknown parameter q_*, element of another (possibly infinite dimensional) Hilbert space X, simultaneously to the evolution of the physical state y from observations of the latter. Even if the dynamical system is linear, this inverse problem typically involves a nonlinear parameter-to-output-map. To our knowledge, theory based online estimation techniques were only analyzed in the context of linear and finite dimensional problems $y(t) = A(t)q_*$. We present an online parameter estimator for nonlinear dynamical problems based on the minimization of a prediction error and utilize Lyapunov theory for its analysis. Numerical results are presented.

1 Introduction

Many time-dependent processes in scientific and technical applications are described by nonlinear dynamical systems of the form

$$\begin{aligned} \dot{y} &= f(q_*, y, t), \\ y(0) &= y_0 \in Y \end{aligned} \tag{1.1}$$

for the physical state y. We shall assume that for any time $t \in R^+$, $y(t)$ belongs to some Hilbert space Y, i.e., that the trajectory y of (1.1) belongs to $\mathcal{Y} = C(R_+, Y)$, the set of continuous mappings from R_+ to Y. The abstract formulation (1.1) might represent a system of ordinary differential equations, i.e., $y(t) \in Y = R^m$, or a system of partial differential equations with $\dim(Y) = \infty$. The underlying model typically involves a (set of) parameter(s) q_* that could either represent a vector of real numbers, i.e., $q_* \in X = R^n$, or that could be (a) function(s) of space variables, the physical state y itself, ..., i.e., q_* could belong to some infinite dimensional Hilbert space X.

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For instance, let us consider the longitudinal aircraft model taken from [18]

$$\begin{aligned}
\dot{V} &= \frac{T\cos(\alpha) - D(\alpha, b)}{m} - g\sin(\theta - \alpha), \\
\dot{\alpha} &= b - \frac{T\sin(\alpha) + L(\alpha, b)}{mV} + g\cos(\theta - \alpha), \\
\dot{b} &= \frac{M(\alpha, b)}{I}, \\
\dot{\theta} &= b,
\end{aligned} \tag{1.2}$$

with true airspeed V, angle of attack α, body pitch rate b, pitch angle θ, mass m, inertia I, acceleration g due to gravity and engine net thrust T. Since lift L, drag D and pitching moment M are not just real numbers but functions of the state variables α and b, the system is nonlinear and transforms into (1.1) with $y = [V, \alpha, b, \theta]^T \in R^4$ and $q_* = [L, D, M]^T$ out of some proper infinite dimensional parameter space.

While the general type of the right-hand side f is usually fixed in the modeling phase, initially unknown parameters q_* involved still have to be adjusted by data related to the physical state y. Depending on the application, there are two options for their estimation. If the available initial guess q_0 is not required to be immediately improved, it is preferable to observe y over a period of time, say $[0, T]$, and to identify q_* afterwards using the data set collected. However, in many situations, such as tuning parameters needed for the control and operation of vehicles and aircrafts, it is not possible to fall back on such offline estimation techniques (i.e., to wait until the aircraft has landed or crashed). Instead, it is necessary to estimate the parameters online, i.e., to update the initial guess q_0 simultaneously to the time evolution of the physical state y. Since y usually depends on q_* in a nonlinear way, even if f in (1.1) is linear with respect to its arguments, both the offline and the online estimation problem belong to the class of nonlinear inverse problems.

For solving the nonlinear offline estimation problem, one can rest on a wide range of methods discussed in the literature. Denoting the observations of y over a period of time by $z(t_1), ..., z(t_m)$ (or a continuous analogue) and introducing the operator F mapping a parameter q onto the solution of (1.1) evaluated at times $t_1, ..., t_m$, most of these methods are related to the minimization of the output least squares term

$$\|z - F(q)\|_Z^2. \tag{1.3}$$

Turning to online estimation problems, the literature only gives advise in context of finite-dimensional linear problems, i.e., for the identification of the vector $q_* \in R^n$ in

$$y(t) = A(t)q_* \tag{1.4}$$

with a time-dependent $m \times n$ matrix $A(t)$, see [17], [16], obtained from applying the Laplace transformation to linear dynamical systems with constant parameters. To our knowledge, nonlinear online estimation problems such as finding $q_* \in X$, $\dim(X) = \infty$, in a (possibly) nonlinear system (1.1) as $y(t)$ evolves have so far not been treated. Though the dynamic inversion of nonlinear maps has also been studied in [8], this technique in general does not apply to parameter estimation because of the (in this context) restrictive assumption $X = Y = R^n$, i.e., parameter and state space coincide (even for $X = Y$ the method would be totally different from our approach).

In this paper, we present an approach to nonlinear online parameter estimation in (possibly infinite dimensional) Hilbert spaces, then still including the finite dimensional special case of identifying a set of real numbers in a nonlinear system of ordinary differential equations. Our main emphasis lies on formalizing the problem on an abstract operator level and on presenting a convergence result for our solution strategy. Our results may then serve as a basis for the development of further online methods including such that are especially tailored for the ill-posed case where the solution does not depend continuously on the data.

In Section 2, we briefly quote the basic concepts of Lyapunov theory that can be utilized for the analysis of dynamical systems and their behaviour around an equilibrium. Section 3 deals with the nonlinear offline estimation problem (1.3) and introduces iterative and continuous methods for its solution. In Section 4, we turn our attention to the simultaneous estimation of the parameter in (1.1) to the evolution of the physical state. We formulate the problem by means of an nonlinear prediction operator and present an online parameter update law based on comparing the observed physical state to the predicted one. Under the assumption that the sought parameter q_* does not depend explicitly on time, we are able to build a dynamical system for the parameter error, whose convergence properties can be analyzed by application of Lyapunov theory. Eventually, numerical results are presented.

2 Fundamentals of Lyapunov Theory in Banach Spaces

The central assertion of Lyapunov theory is that the existence of a generalized energy function V, that is, roughly speaking, positive and decreasing along the system trajectory, allows to infer stability of an underlying dynamical system. Stability aspects based on Lyapunov theory in finite dimensions are widely discussed in the literature, see, e.g., [9]. However, those results do not apply one-to-one if X is infinite dimensional since (among other things) the compactness of bounded sets is lost. In the following, we briefly present the basic definitions and stability results for the infinite dimensional case taken from [20].

Let X be a (possibly) infinite dimensional Banach space and let $\mathcal{D}$ be an open or closed subset containing 0. The distance between an element $e \in X$ and $\mathcal{D}$ is defined as

$$d_{\mathcal{D}}(e) = \inf \{ \|e - x\|, \, x \in \mathcal{D} \} ,$$

$\mathcal{B}_\rho(e)$ denotes the open ball of radius ρ centered at e. For a map H from $\mathcal{D} \times R^+ \to X$, we consider a general dynamical system

$$\begin{aligned} \dot{e}(t) &= H(e(t), t), \;\; t \geq t_0 \\ e(t_0) &= e_0 \in \mathcal{D}, \end{aligned} \tag{2.1}$$

where $e(t) \in \mathcal{D}$ denotes the state of the system and $\dot{e}(t)$ is the time derivative, and we assume

Assumption 1. *1. $\forall t \in R^+$, $H(0, t) = 0$,*

2. $\forall (e_0, t_0) \in \mathcal{D} \times R^+$, the system admits a unique solution $e(t; e_0, t_0)$,

3. the additivity of motions, i.e., $\forall 0 \leq t_0 \leq t_1 \leq t_2$, we have

$$e(t; e(t_2; e_0, t_0), t_2) = e(t; e(t_1; e_0, t_0), t_1) = e(t; e_0, t_0), \;\; t \geq t_2.$$

In particular, Assumption 1 requires $e(t) \in \mathcal{D}$ for all $t \geq t_0$. If H does not depend explicitly on t, i.e., $H = H(e)$, the system is called autonomous, otherwise it is called non-autonomous.

In Lyapunov theory, stability has to be understood in the following sense.

Definition 2.1. *A closed set $D \subset X$ is called an invariant set with respect to* (2.1) *if*

$$\forall e_0 \in D \, \forall t_0 \in R^+ \, \forall t \geq t_0 \;:\; e(t; e_0, t_0) \in D.$$

Definition 2.2. *An invariant set D of* (2.1) *is called uniformly stable (in the sense of Lyapunov) if*

$$\forall R > 0 \, \exists r > 0 \, \forall e_0 \in D \;:\; d_D(e_0) < r \Rightarrow \forall t \geq t_0 \; d_D(e(t, e_0, t_0)) < R.$$

Definition 2.3. *A set D is called uniformly asymptotically stable with respect to* (2.1) *if it is uniformly stable and if it is attractive, i.e., if*

$$\exists r > 0 \, \forall e_0 \in D \;:\; d_D(e_0) < r \Rightarrow \lim_{t \to \infty} d_D(e(t; e_0, t_0)) = 0.$$

Note that attractivity of a set does not necessarily imply its uniform stability.

In the given context, desirable properties that a function may have are:

Definition 2.4. *Let $D \subset \mathcal{D} \subset X$ be a subset and V be a function from $\mathcal{D} \times R^+ \to R^+$. $V(e, t)$ is called positive definite on $\mathcal{D} \backslash D \times R^+$ if*

1. *$\forall(e,t) \in D \times R^+ \,:\, V(e,t) = 0$, and*
2. *$\forall(e,t) \in \mathcal{D} \backslash D \times R^+ \,:\, V(e,t) > 0$.*

The next theorem states the criteria necessary for uniform stability of (2.1):

Theorem 2.1. *Let Assumption 1 hold. Let $\mathcal{K}$ be the set of strictly increasing continuous functions $\beta : R^+ \to R^+$ with $\beta(0) = 0$ and $\lim_{s\to\infty} \beta(s) = \infty$. Let $D \subset \mathcal{D}$ be an invariant set of the system* (2.1)*. If there exists a function $V : \mathcal{D} \times R^+ \to R^+$ called Lyapunov function that satisfies*

1. *V is positive definite on $\mathcal{D}\backslash D$,*
2. *there exists $\beta_1, \beta_2 \in \mathcal{K}$ such that*
$$\beta_1(\rho_D(e)) \le V(e,t) \le \beta_2(\rho_D(e)), \quad \forall(e,t) \in \mathcal{D} \times R^+$$
3. *for all $0 \le t_1 \le t_2$, $V(e(t_2), t_2) \le V(e(t_1), t_1)$,*

then the system (2.1) *is uniformly stable with respect to the invariant set D.*

Hence, the function V is required to be positive definite, bounded and non-increasing along the trajectory. If V is differentiable, then

$$\frac{d}{dt}V(e(t),t) = \langle V_e(e(t),t), \dot{e}(t)\rangle + \frac{\partial V}{\partial t}V(e(t),t) \le 0$$

is a sufficient condition for the latter property. Here, $V_e(\cdot, t) : D \to X^*$ denotes the partial Fréchet derivative of V with respect to e and X^* is the dual to X.

In order to guarantee uniform asymptotic stability, the assumptions on V have to be strengthened.

Theorem 2.2. *Let the system* (2.1) *be uniformly stable with respect to the invariant set D. Then,* (2.1) *is uniformly asymptotically stable with respect to D if and only if the following condition is satisfied*

1. *There exist $\gamma \in \mathcal{K}$ and $\delta > 0$ such that for any $e_0 \in \mathcal{D}$ with $\rho_D(e_0) \le \delta$ there is a strictly increasing time sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that*
$$V(e(t_{n+1}), t_{n+1}) \le V(e(t_n), t_n) - \gamma(\rho_D(e(t_n))), \quad \forall n \ge 0. \tag{2.2}$$

Note that Theorem 2.2 states both sufficient and necessary conditions for uniform asymptotic stability. If V is partially differentiable, the proof given in [20] shows that (2.2) can be replaced by

$$\dot{V}(e(t),t) := \frac{d}{dt}V(e(t),t) \le -\gamma(\rho_D(e(t)))$$

in order to guarantee asymptotic stability.

We emphasize that $\rho_D(e)$ simply reduces to $\|e\|$ in the Theorems stated above for the case $D = \{0\}$ which will be the relevant one for our purposes.

3 Offline Estimation

This section briefly sketches how the problem of offline estimation of the parameter q_* in the dynamical system (1.1) can be treated as nonlinear operator equation.

From now on, let X and U be real Hilbert spaces and Q be a subset of X. Dealing with an offline estimation problem, we furthermore introduce the space $\mathcal{Y}_T = C([0,T], Y)$ for some fixed positive time T. Then, for a parameter $q \in Q$ (for simplicity we assume that there is no explicit time dependency of q) we consider the time evolution of a physical state $y(t) \in Y$ described by the dynamical system

$$\begin{aligned} \dot{y} &= f(q, y, t) \\ y(0) &= y_0 \in Y. \end{aligned} \tag{3.1}$$

We suppose that the map $f : Q \times Y \times R^+ \to Y$ is such that

Assumption 2. *For any parameter $q \in Q$, the direct problem* (3.1) *admits a unique solution $y \in \mathcal{Y}_T$.*

As *inverse problem* we consider in this section the offline identification of the parameter q in (3.1) from observations of the physical state over a time interval $[t_1, t_m] \subseteq [0, T]$, i.e., we only start the estimation process after the data

$$z = (y(t_1), y(t_2), ..., y(t_m)) \in Z = Y^m \tag{3.2}$$

have been collected. By means of the nonlinear forward operator

$$F : Q \to Z,\ q \to (y_q(t_1), y_q(t_2), ..., y_q(t_m)),$$

where the notation y_q emphasizes the dependency of the solution of (3.1) on q, the inverse problem can be formulated as

$$F(q) = z. \tag{3.3}$$

Assuming that the data $z \in Z$ are attainable and contain enough information, the task is to find the parameter $q_* \in Q$ such that $y_{q_*}(t_i) = z(t_i)$, $i = 1, ..., m$.

Abstract nonlinear inverse problems of the form (3.3) have been widely studied, we refer to the monographs [5], [1], [15]. Regularization methods for their stable solution can be roughly divided into Tikhonov type, see [6], [1], and iterative regularization methods, see [7]. The latter are based on an iterative minimization of

$$q \to \frac{1}{2}\|z - F(q)\|^2 \tag{3.4}$$

over Q starting from an initial guess $q_0 \in Q$, well-analyzed examples are the Landweber method

$$q_{k+1} = q_k + F_q(q_k)^*(z - F(q_k)), \tag{3.5}$$

see [11], and the Levenberg-Marquardt method

$$q_{k+1} = q_k + (F_q(q_k)^* F_q(q_k) + \alpha_k I)^{-1} F_q(q_k)^* (z - F(q_k)), \tag{3.6}$$

see [10]. Here, $F_q(q_k)^*$ denotes the adjoint operator of the Fréchet derivative $F_q(q_k)$ of F evaluated at q_k, α_k is a sequence of regularization parameters. Both for (3.5) and (3.6), (local) convergence of the iterates towards a solution q_* of (3.3), i.e.,

$$q_k \to q_* \text{ as } k \to \infty, \tag{3.7}$$

is proven if

$$\|F_q(q)\| \leq M$$

in a neighborhood of q_* and if

$$\|F(\tilde{q}) - F(q) - F_q(q)(\tilde{q} - q)\| \leq \eta \|F(\tilde{q} - F(q)\|, \quad \eta < 1/2, \tag{3.8}$$

for $\tilde{q} - q$ sufficiently small, which can be seen as a restriction on the nonlinearity of F. Other iterative methods for inverse problems, for which a convergence analysis is available, e.g., [2], require

$$\|F_q(q) - F_q(\tilde{q})\| \leq L\|q - \tilde{q}\|$$

and even stronger conditions on F than (3.8).

If the inverse problem (3.3) is ill-posed, i.e., if the solution does not depend continuously on the data z, special care has to been taken if instead of z only an approximation z^δ (then not necessarily attainable) with

$$\|z - z^\delta\| \leq \delta \tag{3.9}$$

is given. In order to overcome the instability, the central idea of regularization is to replace (3.3) by a family of neighboring well-posed problems. In the context of iterative methods, this leads to stopping rules in dependence of the quality of the data, i.e., only if, e.g., (3.5) or (3.6) are stopped at some index $k_*(\delta, z^\delta)$ (with $k_* \to \infty$ as $\delta \to 0$), q_{k_*} can be a reliable approximation to q_*. Since our discussion of offline methods only shall serve setting up the stage for online methods, we stay here with the assumption of exact data.

Iterative methods are tightly coupled to so-called continuous regularization methods. For instance, the initial value problem

$$\begin{aligned} \dot{q}(\tau) &= F_q(q(\tau))^*(z - F(q(\tau))), \\ q(0) &= q_0 \in Q, \end{aligned} \tag{3.10}$$

can be understood as a continuous analogue of the Landweber method (3.5) that is obtained by applying a forward Euler method to (3.10). Here, $\dot{q}$ denotes the derivative with respect to the artificial time variable τ. The continuous Landweber method (3.10) has been analyzed in [19] using similar techniques as in [11], as a consequence, convergence

$$q(\tau) \to q_* \text{ as } \tau \to \infty \tag{3.11}$$

has again been established under the nonlinearity condition (3.8). For another example, we refer to [12], where the continuous regularized Gauss-Newton method is studied based on the results obtained in [2] for its discrete version.

As a first reference to Section 2, we mention the possibility of analyzing continuous regularization methods using Lyapunov theory. There, the central idea is to find a Lyapunoy function $V(e(\tau), \tau)$ for the error $e(\tau) = q(\tau) - q_*$ such that the equilibrium 0 of the system for $e(t)$, e.g.,

$$\begin{aligned} \dot{e}(\tau) &= F_q(e(\tau) + q_*)^*(z - F(e(\tau) + q_*)), \\ e(0) &= q_0 - q_* \in Q, \end{aligned} \tag{3.12}$$

is uniformly asymptotically stable and hence implies (3.11). Still, the link of that approach to the techniques used so far has to be further exploited.

Given the special data structure (3.2), one could also estimate the parameter q_* in (3.1) by considering the sequence of operator equations

$$F(q, t_i) = z_i, \quad i = 1, ..., m, \tag{3.13}$$

with the nonlinear maps $F(\cdot, t_i)$ defined as

$$F(\cdot, t_i) : Q \to Y, q \to y_q(t_i). \tag{3.14}$$

Given an initial guess $q_0 \in Q$, the idea now is to process the data z_i sequentially before updating the parameter in an outer cycle. For instance, the Landweber-Kaczmarz method reads as

$$\begin{aligned} & q_{k,1} = q_k \\ & \quad \text{for } i = 1 : m \\ & \qquad q_{k,i+1} = q_{k,i} + F_q(q_{k,i}, t_i)^*(z_i - F(q_{k,i}, t_i)) \\ & \quad \text{end} \\ & q_{k+1} = q_{k,m}. \end{aligned} \tag{3.15}$$

For infinite dimensional problems, convergence (3.7) has been proven in [13] under assumption (3.8) with F replaced by $F(\cdot, t_i)$ for $i = 1, ..., m$. Given an infinite series of data z_i, i.e., $m \to \infty$, the Kaczmarz method would only iterate on the inner level. Then, to some extent, (3.15) could be understood as a first example for an online estimation method, where i indicates the discrete time variable, see the next section.

For completeness, we finally mention the alternative of so-called all-at-once approaches for solving the offline estimation problem: while all the techniques mentioned above are based on elimination of the state variable y, i.e., (3.4) is only minimized over the parameter space Q, one also might consider the minimization of

$$\frac{1}{2}\|z - (y(t_1), ..., y(t_m))^T\|^2$$

over the product space $Q \times Y$ subject to the constraints (3.1), see [3]. However, our approach for solving the online estimation problem to be discussed in the next section seizes the state elimination concept.

4 Online Estimation

From now on, we focus on estimating the unknown parameter from observations of the physical state simultaneously to the evolution of the latter. Still assuming that the sought parameter does not depend explicitly on time, i.e., $q_* \in Q$, the overall goal is to find an estimator q in

$$\mathcal{Q} = C(R^+, Q)$$

such that $q(0) = q_0 \in Q$ and

$$q(t) \to q_* \text{ as } t \to \infty \tag{4.1}$$

based on comparing the observed state to the one predicted by $q \in \mathcal{Q}$.

For that purpose, we have to slightly change the notion of solvability of the underlying dynamical system, i.e., we now assume that the map $f : \mathcal{Q} \times Y \times R^+ \to Y$ (note the replacement of Q by $\mathcal{Q}$) satisfies

Assumption 3. *For any estimator $q \in \mathcal{Q}$, the dynamical system*

$$\begin{aligned} \dot{y} &= f(q, y, t) \\ y(0) &= y_0 \in Y \end{aligned} \tag{4.2}$$

admits a unique solution $y_q \in \mathcal{Y} = C(R^+, Y)$.

Compared to Assumption 2, explicit time variation of the estimator q now is (and in fact has to be) allowed, furthermore the trajectory y now is defined on the whole positive time axes. Still, the sought parameter q_* is assumed to be not explicitly time-dependent, the existence of the true physical state is covered by Assumption 3 with the special parameter choice $q(t) = q_*$. We emphasize again that q_* may still depend on state, space variable or external quantities.

For a given estimator $q \in \mathcal{Q}$, we call the solution $y_q \in \mathcal{Y}$ of (4.2) the predicted state, the associated mapping is formalized by the nonlinear operator

$$F : \mathcal{Q} \to \mathcal{Y}, q \to y_q.$$

Regarding the concatenation of F and the time evaluation operator

$$T(t) : \mathcal{Y} \to Y, y \to y(t),$$

we eventually define the *nonlinear prediction operator*

$$F(\cdot, t) : \mathcal{Q} \to Y, q \to y_q(t),$$

compare to (3.14). Note that for the evaluation of $F(q, t)$ the future values of $q \in \mathcal{Q}$, i.e., $q(\tau)$ with $\tau > t$, are irrelevant.

Observing the development of the true physical state y_{q_*}, i.e., given $z(\tau) = y_{q_*}(\tau)$ for $\tau \leq t$, where t denotes the current observation time, the task in online estimation is to drive the parameter estimator q in the sense of (4.1). The minimization of the squared prediction error

$$\frac{1}{2}\|z(t) - F(q, t)\|_Y^2 \tag{4.3}$$

at time t may serve as a basis for the construction of the parameter estimation method. However, in order to obtain an online method the derivative of (4.3) may not be build with respect to all values of q from time 0 up to t since changes of the estimator for past values would require to restart the prediction process, i.e., the evaluation of $F(q, t)$, again and again. Instead, we only vary (4.3) with respect to the current value $q(t)$ of the estimator and introduce the *linearized prediction operator*

$$F'(q, t) : Q \subset X \to Y,\ p \to w(t),$$

where w solves the linearized state equation

$$\begin{aligned} \dot{w} &= f_y(q, y_q, t)w + y_q(q, y_q, t)p, \\ w(0) &= 0. \end{aligned} \tag{4.4}$$

As opposed to the standard derivative of $F(\cdot, t) : \mathcal{Q} \to Y$ the variation p in (4.4) only belongs to Q but not to $\mathcal{Q}$. In the following we assume that the linear operator $F'(q, t)$ is continuous (for any t and in some neighborhood of q_*), i.e.,

$$\|F'(q, t)\| \leq M,\ q \in C(R^+, \mathcal{B}_\rho(q_*)). \tag{4.5}$$

As a consequence, the operator $F'(q, t)$ acting between the Hilbert spaces X and Y admits an adjoint operator $F'(q, t)^*$.

Accepting the presented linearization concept, a first idea for constructing an online estimator based on (4.3) could be the gradient flow

$$\begin{aligned} \dot{q}(t) &= \lambda F'(q,t)^*(z(t) - F(q,t)), \\ q(0) &= q_0 \in Q. \end{aligned} \tag{4.6}$$

First numerical tests (for the example to be presented in Section 6) indicated a rather slow convergence of method (4.6). Based on techniques discussed for the linear problem (1.4) in [17], we replace the gain factor λ by a time-dependent linear gain operator $G(t) : X \to X$ and consider the online estimation method

$$\begin{aligned} \dot{q}(t) &= G(t)F'(q,t)^*(z(t) - F(q,t)), && (4.7)\\ q(0) &= q_0 \in Q, \\ \dot{G}(t) &= \alpha(G(t) - G(t)\underline{G}^{-1}G(t)) - G(t)F'(q,t)^*F'(q,t)G(t), && (4.8)\\ G(0) &= G_0. \end{aligned}$$

Hence, the method which requires to simultaneously solve the equations for the estimator q, for the predicted state (evaluation of $F(q,t)$) and for the linearized state (4.4) is complemented by a dynamical equation for G starting from a positive definite initial operator $G_0 : X \to X$ with

$$(G_0 x, x) \leq g_0 \|x\|^2, x \in X,$$

or $G_0 \leq g_0 I$ in simplified notation, for some positive constant g_0. Equation (4.8), in which α is a positive constant and the positive definite operator $\underline{G}^{-1} : X \to X$ acts as a lower bound for $G^{-1}(t)$ (we choose $\underline{G}^{-1} \geq \frac{1}{\underline{g}} I$ with $\underline{g} > g_0$), is motivated by the minimization in [17] of a total weighted prediction error

$$\int_0^t e^{-\alpha(t-\tau)} \|z(\tau) - A(\tau)q(t)\| d\tau$$

for the linear problem (1.4). However, it is so far not clear if (4.8) can be linked to a similar error concept for the nonlinear case, maybe a time dependent $\alpha(t)$ in (4.8) with $0 < \underline{\alpha} \leq \alpha(t) \leq \overline{\alpha}$ helps for that purpose which in fact is allowed by the proof of Theorem 5.1 to be presented below.

In the next section we show that the desired convergence result (4.1) is obtained for the online method (4.7) under the assumption

$$\exists \gamma,\, \beta > 0 \;\forall t \in R^+ \quad \int_t^{t+\beta} F'(q,\tau)^* F'(q,\tau) d\tau \geq \gamma I \tag{4.9}$$

(again for some neighborhood of q_*). Basically, this condition, based on the linear theory in [17], [16], states that the linearized physical state corresponding to the parameter estimator does not decay to zero too fast. Another rough interpretation of (4.9) is that the physical state y_{q_*} has to be sufficiently excited by the system input such that the observed data z contain enough information for the identification of the parameter q_*. As we will demonstrate in our numerical example, the time dependency of f in (4.2) plays a central role in that context.

5 Convergence Analysis

In this section, we utilize the results of Section 2 in order to show that the parameter error

$$e(t) = q(t) - q_* \tag{5.1}$$

asymptotically converges to the origin. For that purpose, we assume that the solution $q_* \in Q$, i.e., $y_{q_*} = z$, is locally unique, and that the initial guess $q_0 \in Q$ lies sufficiently close to q_*. Hence, the initial error $e(0) = q(0) - q_*$ belongs to some neighborhood $\mathcal{B}_\rho(0)$ of the origin. Besides of assumptions (4.5) and (4.9) we suppose for our analysis that the operators $F(\cdot,t)$ and $F'(q,t)$ satisfy

$$\begin{aligned} & \alpha\underline{c}(t)\langle e(t), e(t)\rangle + \langle F'(q,t)e(t), F'(q,t)e(t)\rangle \\ \geq\ & 2\langle y(t) - F(q,t) + F'(q,t)e(t), F'(q,t)e(t)\rangle + \kappa(\|e(t)\|), \end{aligned} \tag{5.2}$$

for some $\kappa \in \mathcal{K}$ and $\underline{c}$ defined in (5.6). Since (5.2) would be satisfied by a linear operator, we refer to (5.2) as a condition that restricts the nonlinearity of $F(\cdot,t)$ (compare to (3.8)). Both (5.2) and (4.9) are formulated as actually needed in the forthcoming analysis. Since q and q_* are not known in advance, the conditions are in fact assumed to hold in a neighborhood of q_*, i.e., for $q \in C(R^+, \mathcal{B}_\rho(q_*))$.

Theorem 5.1. *Let* (4.9)*,* (5.2)*,* (4.5) *hold in a neighborhood of the locally unique solution* q_**. Then, the parameter estimator* q *defined according to* (4.7)*,* (4.8) *satisfies* $q(t) \to q_*$ *as* $t \to \infty$*.*

Proof. We consider the dynamical system

$$\begin{aligned} \dot{e}(t) &= G(t)F'(e+q_*,t)^*(z(t) - F(e+q_*,t)), \\ e(0) &= q_0 - q_*, \\ \dot{G}(t) &= \alpha(G(t) - G(t)\underline{G}^{-1}G(t)) - G(t)F'(e+q_*,t)^*F'(e+q_*,t)G(t), \\ G(0) &= G_0 \end{aligned} \tag{5.3}$$

for the error (5.1) obtained from (4.7). Under the assumptions made, we next show that

$$V(e,t) = (G^{-1}(t)e, e) \tag{5.4}$$

is a proper candidate for the Lyapunov function in Theorem 2.2. First, one observes from differentiating the identity $G^{-1}(t)G(t) = I$ with respect to t, that

$$\dot{G^{-1}}(t) = -\alpha(G^{-1}(t) - \underline{G}^{-1}) + F'(e+q_*,t)^*F'(e+q_*,t).$$

Hence, it follows

$$G^{-1}(t) = (G_0^{-1} - \underline{G}^{-1})e^{-\alpha t} + \underline{G}^{-1} + \int_0^t e^{-\alpha(t-r)} F'(q,r)^* F'(q,r) dr.$$

Now, persistence of excitation (4.9) allows to improve the lower bound $\underline{G}^{-1}$ on $G^{-1}(t)$ according to

$$G^{-1}(t) \geq \underline{G}^{-1} + \begin{cases} (G_0^{-1} - \underline{G}^{-1})e^{-\alpha\beta} & t < \beta \\ e^{-\alpha\beta}\gamma I & t \geq \beta \end{cases},$$

i.e.,

$$G^{-1}(t) \geq \underline{G}^{-1} + \underline{c}(t) I \tag{5.5}$$

with

$$\underline{c}(t) = \begin{cases} (\frac{1}{g_0} - \frac{1}{\underline{g}})e^{-\alpha\beta} & t < \beta \\ e^{-\alpha\beta}\gamma & t \geq \beta \end{cases} > 0. \tag{5.6}$$

Regarding an upper bound for $G^{-1}(t)$, we first observe

$$G^{-1}(t) \leq G_0^{-1} + \int_0^t e^{-\alpha(t-r)} F'(q,r)^* F'(q,r) dr.$$

The integral expression solves the stable filter equation

$$\dot{B} + \alpha B = F'(q,t)^* F'(q,t),$$

whose output is bounded due to the boundedness (4.5) of $F'(q,t)$. Hence, there exists some $\overline{c} > 0$ such that

$$G^{-1}(t) \leq G_0^{-1} + \overline{c} I.$$

As a consequence, the candidate (5.4) satisfies the first condition in Theorem 2.1.

Next, we consider the derivative of (5.4) along the trajectory of (5.3) and obtain

$$\begin{aligned}
\dot{V}(e(t),t) &= (\dot{G}^{-1}(t)e(t), e(t)) + 2(G^{-1}(t)\dot{e}(t), e(t)) \\
&= -\alpha((G^{-1}(t) - \underline{G}^{-1})e(t), e(t)) + (F'(q,t)^* F'(q,t)e(t), e(t)) \\
&\quad +2(F'(q,t)^*(z(t) - F(q,t)), e(t)) \\
&= 2(z(t) - F(q,t) + F'(q,t)e(t), F'(q,t)e(t)) - \alpha(G^{-1}(t) - \underline{G}^{-1})e(t), e(t)) \\
&\quad -(F'(q,t)e(t), F'(q,t)e(t)) \qquad (5.7) \\
&\leq -\kappa(\|e(t)\|), \qquad (5.8)
\end{aligned}$$

where we used (5.5) and (5.2). Hence, Theorem 2.2 guarantees convergence of $e(t) \to 0$ as $t \to \infty$ because of (5.8) and the boundedness of G^{-1}. □

Though we cannot give a theoretical justification, the numerical results to be presented next and the linear theory [17] suggest that assumption (4.9) in fact implies (at least local) uniqueness of q_*, i.e., that the explict uniqueness assumption in Theorem 5.1 is redundant. Both (4.9) and (5.2) will be subject of the discussion in Section 6.

6 Numerical Tests

For the sake of straightforwardness of the discussion, we consider the nonlinear ordinary differential equation

$$\begin{aligned} \dot{y}(t) &= 4 - u(t)\sin(y) - q(y) + \cos(\frac{\pi}{2} - y), \\ y(0) &= 0, \end{aligned} \tag{6.1}$$

as underlying state equation, the true parameter for our tests is chosen as

$$q_*(y) = 2 + 3y + y^2. \tag{6.2}$$

The type of nonlinearity in (6.1) reminds of that given in the dynamic aircraft system (1.2), also suggested in [14], which motivates our concentration on (6.1). The explicit time dependency of the right hand side in (6.1) is strongly influenced by the function u (we will refer to u as control function) which is allowed to switch (in a smooth manner) between

$$u_{np}(t) = 3 \tag{6.3}$$

or

$$u_p(t) = 3\sin(0.05t). \tag{6.4}$$

For the first series of tests, we assume to know the polynomial behaviour of the sought parameter q_*, then, with $Q = X = R^3$, the task is to determine the coefficients in (6.2). The corresponding implementation (all numerics were done in MATLAB based on the ode45-solver) of the parameter update law (4.7) leads to a dynamical system for the estimators q_1, q_2, q_3 and the 3×3 gain matrix G. The convergence analysis of Section 5 states that convergence

$$q_1(t) \to 2,\ q_2(t) \to 3,\ q_3(t) \to 1 \text{ as } t \to \infty \tag{6.5}$$

can be expected if condition (4.9) is satisfied, i.e., if the state y_{q_*} is permanently excited such that the corresponding observation z contains enough information. With the linearized predicted state equation given by

$$\begin{aligned} \dot{w} &= -u(t)\cos(y_q)w - q'(y_q)w + \sin(\frac{\pi}{2} - y_q)w - p(y_q) \\ w(0) &= 0, \end{aligned}$$

it is obvious that the left hand side in condition (4.9) is strongly determined by the variability of the control function u. Our results will support the conjecture that the state is sufficiently excited in times when u coincides with (6.4) while it is not in times when u coincides with (6.3).

For our computations, we have chosen the initial parameter guess $q_1(0) = q_2(0) = q_3(0) =$

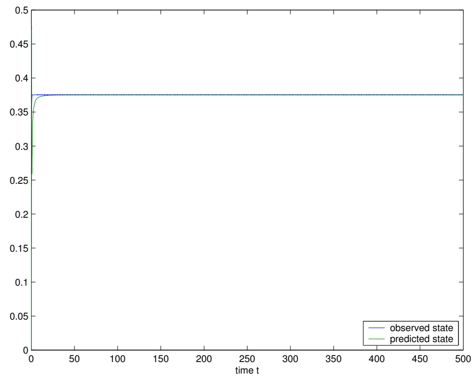

Figure 1: no persistence of excitation

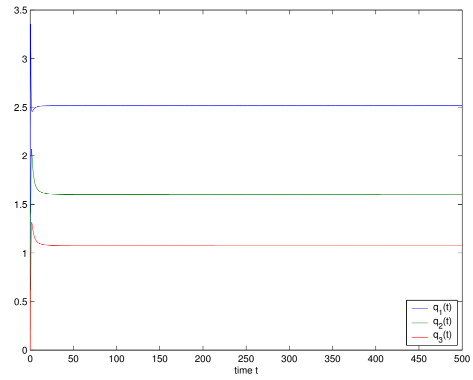

Figure 2: non uniqueness of the solution

0, the initial gain $G(0) = 1500 \cdot I$ (where I denotes the unit matrix), $\alpha = 0.05$ and $\underline{G} = 1000 \cdot I$. Figures 1 and 2 show the performance of the system (4.7), (4.8) if u is chosen according to (6.3). The observed state immediately gets stationary, hence not providing enough information for the identification of the parameter q_*. Nevertheless, the state is predicted correctly, also we have

$$q_1(t) + q_2(t)y_q(t) + q_3(t)y_q(t) = 2 + 3z(t) + z(t)^2 \tag{6.6}$$

for $t > 20$, clearly indicating that for not persistent excitation unique solvability of the inverse problem cannot be expected. This changes if (6.3) is replaced by (6.4) as can be seen in Figures 3 and 4. Under persistence of excitation the parameter estimators converge to the true values (6.5). An adequate speed of convergence can be obtained by choosing the initial gain $G(0)$ sufficiently large. These results are also outlined in a combined test

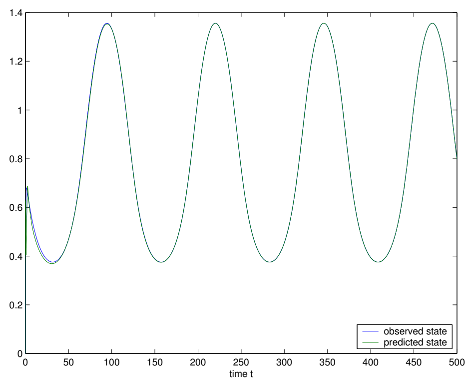

Figure 3: persistence of excitation

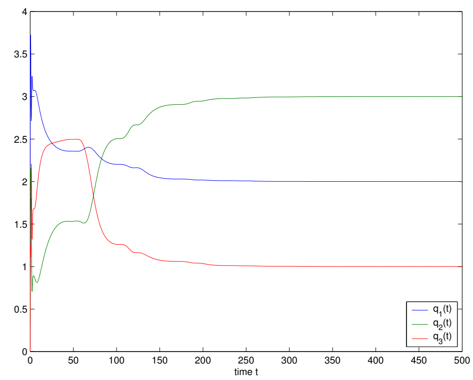

Figure 4: uniqueness of the solution

illustrated in Figures 5 and 6 where u switches from (6.3) to (6.4) at $t = 200$. As soon as

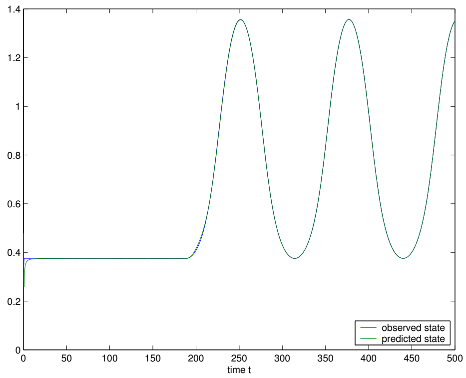

Figure 5: from no persistence of excitation to p.e.

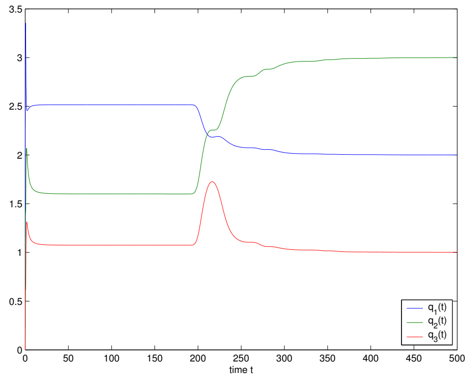

Figure 6: from non uniqueness to uniqueness

the state is persistently excited (at least over a sufficiently long period), the estimators start to converge to the true values. Against this background it is a natural wish to check the persistence of excitation condition (4.9) during the online computations and to assure its satisfaction by - if necessary - adjusting the control function u. The main difficulty lies in the facts that the interval of integration determined by β in (4.9) (and in its counterpart from the linear theory [17]) is usually not known and might be arbitrarily large - if it at all exists. However, there is a necessary condition for (4.9) that can be surveyed online, namely

$$\lambda_{\text{min}} \left(\int_0^t F'(q,\tau)^* F'(q,\tau) d\tau \right) \to \infty \text{ as } t \to \infty, \tag{6.7}$$

where λ_{min} denotes the smallest eigenvalue. Figure 7 indicates that (6.7) is satisfied in case of persistence of excitation, while the plot of $\lambda_{\text{min}}(t)$ corresponding to (6.3) shows that (6.7) and hence (4.9) is not satisfied. The objective of an associated control algorithm would then be to alter u such that (6.7) and (4.9) hold, see also Section 7.

Turning to (5.2), the second crucial assumption in our theory, Figure 8 shows the time derivative of the Lyapunov function (5.4) according to (5.7) (for clarity on different scales) both for the persistently and not persistently excited case. Of course, these plots cannot be generated in practice since they require the unknown solution q_*. Nevertheless, they demonstrate that an adequate function $\kappa \in \mathcal{K}$ in (5.8) exists in case the state is persistently excited, while it does not in the stationary case, and hence support assumption (5.2) of the convergence theory. The initial positive outlier is probably due to the bad initial guess $q(0) = 0$.

In our second series of tests we abandon the a-priori knowledge about the polynomial

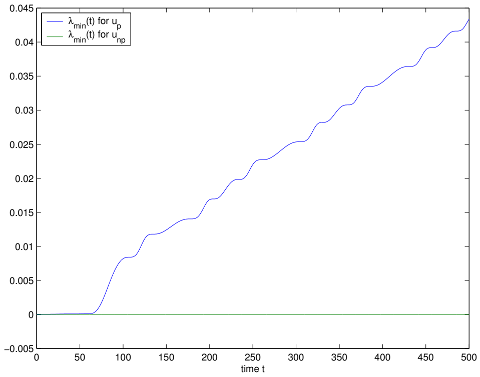

Figure 7: online survey of the necessary condition (6.7)

behaviour of q_*. Now the parameter estimator $q(t)$ is represented via cubic splines, i.e., we use the ansatz

$$q(t, y) = \sum_{i=1}^{m} c_i(t)\varphi_i(y). \tag{6.8}$$

One problem that may arise in online identification of state dependent parameters is that no bounds for the course of y and hence no estimates for the domain of the ansatz functions φ_i in (6.8) are available. One idea might be to adapt this domain during the identification process based on the current observation $z(t)$, however, for our preliminary test we simply have chosen the "sufficiently large" interval $[0, 1.5]$ and $m = 25$. Figures 9 - 11 shows the course of the state prediction and parameter estimation under (6.4) at several points of time (same initial conditions as before). As already expected, the parameter q_* is uniquely recovered on the interval $[0.3755, 1.356]$ that is actually attained by the data $z(t)$ (the bounds between $z(t)$ oscillates). Outside, there is not enough or no information at all available making a reasonable estimation impossible. Note that the interval of relevance is "conquered" by the estimator in accordance to the progress of the data $z(t)$. Figure 12 illustrates the behaviour for the not persistently excited case. As before, the estimator $q(t)$ cannot fully recover the parameter q_*, nevertheless we have $q_*(z(t)) = q(t, y_q(t))$ for $t > 20$, compare to (6.6).

7 Conclusions and Outlook

Estimating parameters in dynamical systems simultaneously to the evolution of the latter is of highest interest in a wide area of applications such as control of vehicles and aircrafts, see [17]. So far, online techniques have only been discussed in the context of finite dimensional and linear dynamical systems. Since a realistic description of real world processes in most cases falls back on nonlinear models in infinite dimensional functional spaces, the

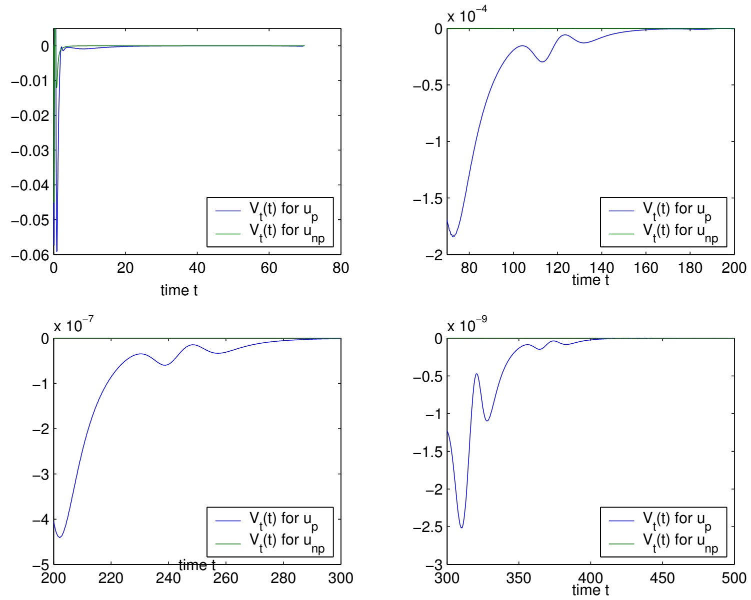

Figure 8: $\dot{V}(e(t),t)$ according to (5.7)

need for the development and analysis of appropriate online estimation methods becomes obvious. The comprehensive theory on offline estimation techniques, also available in nonlinear and infinite dimensional situations, might serve as a valuable source for that purpose.

In this paper, we took a first step towards online parameter estimation for nonlinear problems and presented a parameter update law whose practicability is indicated both by the convergence analysis and the numerical tests. A future goal is to base parameter convergence upon conditions that are practically more revisable (as in the theory for linear and finite dimensional problems). Closely related is the question of how to choose the control function u such that maximal sensitivity of the parameters to the data can be obtained.

Another control related task could be to couple the online estimator to real time control algorithms, e.g., [4], which aim at driving the state y in a desired way by appropriately choosing the control function u. These techniques are again based on state prediction and hence require reliable parameter estimation which so far is still done in an offline way. Note that (4.7), (4.8) could also be understood as an online control algorithm with z denoting the desired state trajectory and q the control parameter.

Future accentuation also has to be laid on the influence of measurement errors, since exact observations of the physical state are in practice not realisable. One idea might be

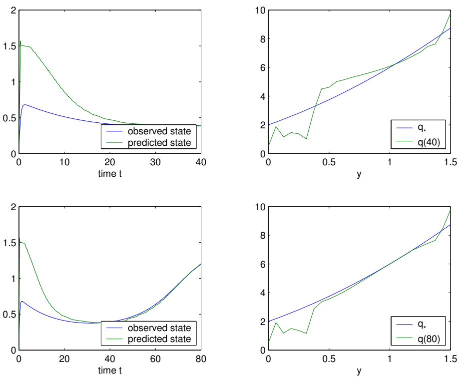

Figure 9: snap-shot of the estimation at $t = 40$ and $t = 80$

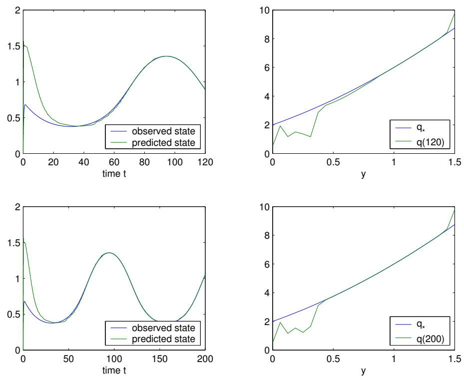

Figure 10: snap-shot of the estimation at $t = 120$ and $t = 200$

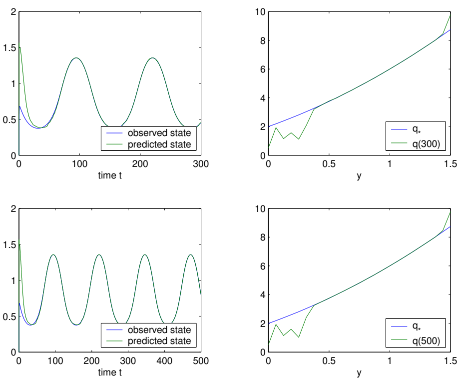

Figure 11: snap-shot of the estimation at $t = 300$ and $t = 500$

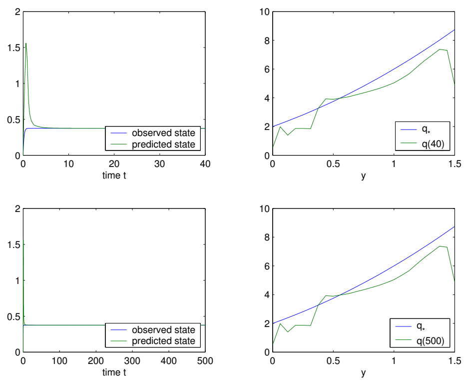

Figure 12: snap-shot of the estimation at $t = 40$ and $t = 500$

to study the parameter dynamics

$$\dot{q}(t) = G(t)F'(q,t)^*(z(t) - F(q,t)) + G(t)F'(q,t)^*(z^\delta(t) - z(t)). \tag{7.1}$$

Then, in the finite dimensional case, Lyapunov theory and the result of Section 5 would imply total stability of q_*, i.e., the solution q to (7.1) stays close to q_* provided the initial error and the data perturbation is sufficiently small, which is also supported by preliminary numerical tests. However, it is still not clear how to regularize the algorithm (4.7), (4.8) (with $z(t)$ replaced by z^δ). Stopping the identification process at some time $t_*(\delta)$ as suggested by the theory for offline problems seems not to be appropriate in the online case since bad future data, i.e., $z^\delta(t)$ for $t > t_*$, are certainly better than no future data at all. Other topics for future research are numerical and theoretical extensions allowing to take explicitly time dependent parameters q_* into account and comparisons to the extended continuous Kalman filter in cases only parts of the state y, i.e., some components or boundary traces,..., can be observed.

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