Multigrid Preconditioned Solvers for Some Elasto-plastic Problems*†

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Abstract

In this paper we discuss an efficient solution method for problems of elasto-plasticity. The phenomenon of plasticity is modeled by an additional term in the stress-strain relation, the evolution of this additional term in time is described by the Prandtl-Reuß normality law. After discretizing the problem in time, we derive a dual formulation. Our solution algorithm is based on an equivalent minimization problem, which is presented for an isotropic hardening law. Since the objective is non-differentiable, we use a differentiable, piecewise quadratic regularization. The algorithm is a successive sub-space optimization method: In the first step, we solve a Schur-complement system for the displacement variable using a multigrid preconditioned conjugate gradient method. The second step, namely the minimization in the plastic part of the strain, is split into a large number of local optimization problems. Numerical tests show the linear complexity of the presented algorithm.

Keywords Elasto-plasticity, finite element method, multigrid method

1 Introduction

The use of elastic material laws in mechanical models is often not sufficient in many real life applications. The phenomenon of plasticity can be described by an additional non-linear term in the stress-strain relation. Plasticity models have a long history in the engineering community. The interested reader is referred to the excellent monographs by Kachanov [13] and Zienkiewicz [22] for detailed information. The rigorous mathematical and numerical analysis of different elasto-plastic models has

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been a topic of mathematical research during the last two decades, see e.g. [9], [10], [14], [20] and the literature cited there.

The admissible stresses are restricted by a yield function depending on the hardening of the material. Furthermore, this yield function characterizes the plastic behavior: isotropic hardening, kinematic hardening, visco-plasticity and perfect hardening. The Prandtl-Reuß normality law describes the time development.

The starting point of the finite element method is the time-discretized variational formulation. This dual formulation in each time step is equivalent to an optimization problem depending only on the displacement vector u and the plastic part of the strain p:

$$f(u,p) = \min_{v,q} f(v,q),$$

under an equality constraint, with f being a convex, non-differentiable function with quadratic terms. Further on, only the case of plasticity with isotropic hardening will be considered. A differentiable and piecewise quadratic objective is obtained by regularization of f, thus standard methods can be applied.

The main idea for the algorithm is to use the Schur-complement form of the discretized problem in the displacement variable u_h. Thus the minimization problem reduces to

$$u_h = \text{argmin}_{v_h} \tilde{f}(v_h, q_{\text{opt}}(v_h)),$$

where $q_{\text{opt}}(v_h)$ denotes the optimal plastic part of the strain with respect to v_h. The system matrix depends nonlinearly on p, therefore the problem is linearized in this variable. Correcting the error, the plastic part of the strain is determined locally by Newton's method.

The method presented in this talk is based on the approach proposed by C. Carstensen [4]. In contrast to [4], we introduce some regularization of the local minimization problems making the cost functional differentiable and allowing us to use the fast converging Newton method. Moreover, we use a special adapted to the problem, multigrid preconditioned conjugate gradient (PCG) method for the Schur-complement problems arising at each incremental step.

The multigrid PCG solver for the elasto-plastic Schur-complement together with above mentioned Newton solver for the local minimization problems allows to solve efficiently large scale 2D and 3D plasticity problems. These features make this approach competitive with other approaches proposed in monographs [9], [14], and in the collection [17] reflecting the state-of-the-art in fast plasticity solvers.

For the theory of elasto-plasticity based on variational inequalities and its analytical background see e.g. [4], [6], [7], [9], [10], [16], [18], [19] and [21]. Multigrid literature can be found in e.g. [2], [8], and [12]. For the theory of finite elements see e.g. [1], [3], and [5].

The paper is organized as follows: In Section 2, we give a brief overview on the basic equations of elasticity and plasticity. First we derive a dual formulation and then an equivalent optimization problem for general hardening laws. Furthermore, only the case of isotropic hardening will be considered. Section 3 is devoted to the construction of the solution algorithm. In Section 4, numerical experiments show the fast convergence and the efficiency of the algorithm. Finally, an outlook on the work still to do is given.

2 Elasto-plasticity

2.1 Definition of the problem

According to the basic theorem of Cauchy, the stress field $\sigma \in L^2(\Omega, \mathbb{R}^{n\times n})$ of a deformed body Ω in $\mathbb{R}^n$ $(n = 2, 3)$ with Lipschitz-continuous boundary has to fulfill the equations

$$\sigma = \sigma^T \quad \text{in } \Omega, \tag{1}$$

$$-\text{div}\, \sigma = b \quad \text{in } \Omega, \tag{2}$$

with b being the vector field of given body forces. The linearized Cauchy-Green strain tensor is appropriate in the case of small deformations, and is obtained by using the displacement vector $u \in H_0^1(\Omega)^n$:

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \quad \text{a. e. in } \Omega. \tag{3}$$

Moreover, in the case of small deformations the strain is split additively into two parts:

$$\varepsilon(u) = \mathbb{A}\,\sigma + p \quad \text{a. e. in } \Omega. \tag{4}$$

Here, $\mathbb{A}\,\sigma$ denotes the elastic, and p the plastic part. The linear, symmetric, positive definite mapping $\mathbb{A}$ from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$ describes the linear elasticity, thus $\mathbb{C} = \mathbb{A}^{-1}$ is the elasticity tensor.

Purely elastic material behavior is characterized by $p \equiv 0$. The modeling of plasticity requires another material law in order to determine p. There are restrictions on the stress variables described by a dissipation functional φ, which is convex, and non-negative, but may also attain $+\infty$. The first restriction is

$$\varphi(\sigma, \alpha) < \infty \quad \text{a. e. in } \Omega. \tag{5}$$

The hardening parameter α is the memory of the considered body and describes previous plastic deformations. Its structure and dimension depend on the hardening law. The above inequality indicates that α controls the set of admissible stresses. The pair (σ, α) is called generalized stresses and values are called admissible if $\varphi(\sigma, \alpha) < \infty$.

The time development of p and α is given by the Prandtl-Reuß normality law which states that for all other generalized stresses (τ, β) there holds:

$$\dot{p} : (\tau - \sigma) - \dot{\alpha} : (\beta - \alpha) \leq \varphi(\tau, \beta) - \varphi(\sigma, \alpha) \quad \text{a. e. in } \Omega, \tag{6}$$

where $\dot{p}$ denotes the time derivative of p, i.e., $\dot{p} = \frac{\partial p}{\partial t}$, and $:$ is the scalar product of matrices such that $A : B = \sum_{i,j=1}^{n} A_{ij} B_{ij}$ for all $A, B \in \mathbb{R}^{n\times n}$.

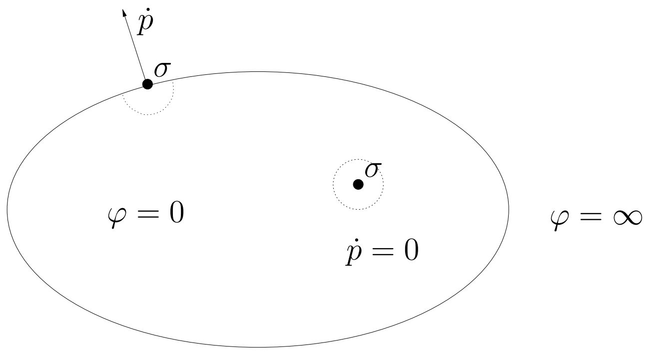

Figure 1: Domain of admissible stresses

If we consider the two above inequalities (5) and (6) without α, that is the case of perfect plasticity, then φ describes the domain where the stress is admissible, see figure 1. The functional φ attains only the values 0 or $+\infty$. Inequality (5) allows σ to remain only within the convex domain. It follows from (6) that in the interior of this domain the time derivative of p has to satisfy the inequality for all τ chosen from the dotted circle, so it has to be zero, and the material behaves elastically. Whereas on the boundary, $\dot{p}$ can only show in the direction of the normal vector for all τ from the dotted semicircle if the material behaves plastically.

Now we are in the position to define the initial value problem:

Problem 2.1. *We look for the displacements $u \in W^{1,2}(0,T;\ H_0^1(\Omega)^n)$, the plastic strain $p \in W^{1,2}(0,T;L^2(\Omega,\mathbb{R}^{n\times n}))$, the stress field $\sigma \in W^{1,2}(0,T;L^2(\Omega,\mathbb{R}^{n\times n}))$, and the hardening parameter $\alpha \in W^{1,2}(0,T;L^2(\Omega,\mathbb{R}^m))$, such that (1) - (6) are satisfied under the initial condition $b(0) = 0$.*

The time dependent variational inequality (6) is solved by an implicit time discretization, for example generalized midpoint rules like Crank-Nicholson or implicit Euler schemes. An implicit Euler discretization for the weak formulation leads to the next problem definition:

Problem 2.2. *Let $H \subseteq H_0^1(\Omega)^n, L_{sym}^{n\times n} \subseteq L^2(\Omega,\mathbb{R}_{sym}^{n\times n})$ and $L^m \subseteq L^2(\Omega,\mathbb{R}^m)$ be closed subspaces. We seek $(u,p,\sigma,\alpha) \in H \times L_{sym}^{n\times n} \times L_{sym}^{n\times n} \times L^m$, such that for given $u_0 \in H$; $p_0, \sigma_0 \in L_{sym}^{n\times n}$ and $\alpha_0 \in L^m$ at some time step t_0 the following conditions are satisfied for $t_1 = t_0 + \Delta t$:*

$$\int_\Omega \sigma : \varepsilon(v)\, dx = \int_\Omega b\, v\, dx \quad \forall v \in H, \tag{7}$$

$$\begin{aligned} &\int_\Omega \{(p-p_0) : (\tau - \sigma) - (\alpha - \alpha_0) : (\beta - \alpha)\}\, dx \\ &\qquad \le \Delta t \int_\Omega \varphi(\tau,\beta)\, dx - \int_\Omega \varphi(\sigma,\alpha)\, dx, \end{aligned} \tag{8}$$

for all $(\tau,\beta) \in L_{sym}^{n\times n} \times L^m$.

$\mathbb{R}^{n\times n}_{sym}$ denotes real, symmetric $n \times n$ matrices. The inequality (8) reads as

$$\frac{1}{\Delta t}(p - p_0, -\alpha + \alpha_0) \in \partial\varphi(\sigma, \alpha). \tag{9}$$

The sub-differential $\partial\varphi(b)$ is defined by the relation

$$a \in \partial\varphi(b) \Leftrightarrow \int_\Omega a : (c - b)\, dx \le \int_\Omega \varphi(c)\, dx - \int_\Omega \varphi(b)\, dx.$$

Since φ is convex, the above equation is equivalent to:

$$(\sigma, \alpha) \in \partial\varphi^*(\frac{1}{\Delta t}(p - p_0, -\alpha + \alpha_0)). \tag{10}$$

φ^* is the dual functional of φ, which is computed by the Fenchel transformation: $\varphi^*(y) := \sup_x \{y : x - \varphi(x)\}$.

Substituting $\sigma = \mathbb{C}\,(\varepsilon(u) - p)$ in (7) and (10) the simplified Problem 2.3 can be obtained.

Problem 2.3. *Find a triple* $(u, p, \alpha) \in H \times L^{n\times n}_{sym} \times L^m$*, such that the following conditions are satisfied for all* $(v, q, \beta) \in H \times L^{n\times n}_{sym} \times L^m$*:*

$$\int_\Omega \mathbb{C}[\varepsilon(u) - p] : \varepsilon(v) dx = \int_\Omega b\, v\, dx, \tag{11}$$

$$\begin{aligned} \int_\Omega \{\mathbb{C}[\varepsilon(u) - p] : (\Delta t\, q - p + p_0) + \alpha : (-\Delta t \beta + \alpha_0 - \alpha)\} dx \le \\ \le \Delta t \int_\Omega \varphi^*(q, \beta) dx - \Delta t \int_\Omega \varphi^*(\frac{p - p_0}{\Delta t}, \frac{\alpha_0 - \alpha}{\Delta t}) dx. \end{aligned} \tag{12}$$

Problem 2.3 is the stationary condition of a minimizer in the following minimization problem 2.4. Vice versa, a minimizer of f of Problem 2.4 is a solution of 2.3:

Problem 2.4. *Find the minimizer* $(u, p, \alpha) \in H \times L^{n\times n}_{sym} \times L^m$ *of*

$$\begin{aligned} f(u, p, \alpha) := \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u) - p] : (\varepsilon(u) - p) dx + \frac{1}{2}\int_\Omega |\alpha|^2 dx \\ + \Delta t \int_\Omega \varphi^*(\frac{p - p_0}{\Delta t}, \frac{\alpha_0 - \alpha}{\Delta t}) dx - \int_\Omega b\, u\, dx. \end{aligned} \tag{13}$$

The dissipational functional φ and thus its dual functional φ^* are determined by the special kind of hardening law: isotropic hardening, kinematic hardening, visco-plasticity and perfect hardening. For deriving an algorithm, the dual functional has to be calculated explicitely. From now on, only the case of isotropic hardening will be considered. But the structure of the optimization problems are similar for the other hardening laws, so the basic idea of the algorithm will work again and will lead to slight modifications in the algorithm.

2.2 Isotropic hardening

In the case of isotopic harding the space dimension m of the hardening parameter α is 1, i.e., α is a scalar function. The dissipational functional φ is defined as follows (see [4]):

$$\varphi(\sigma,\alpha) = \begin{cases} 0 & \text{if } \Phi(\sigma,\alpha) \leq 0, \\ \infty & \text{if } \Phi(\sigma,\alpha) > 0, \end{cases}$$

with the given yield function $\Phi(\sigma,\alpha) = |\operatorname{dev}\sigma| - \sigma_y(1+\alpha H)$. $\sigma_y > 0$ is the initial yield stress and $H > 0$ the modulus of hardening. dev is the deviation and defined by $\operatorname{dev} A := A - \frac{1}{n}\operatorname{tr}(A)\cdot I_n$, where $\operatorname{tr}(A) := \sum_{i=1}^n A_{ii}$ is the trace of a matrix.

The dual functional will be computed only for values from the set of admissible stresses, i.e., $\varphi(\sigma,\alpha) = 0$:

$$\varphi^*(A,B) = \sup_{|\operatorname{dev}\sigma| \leq \sigma_y(1+\alpha H)} (\sigma : A + \alpha B)$$

The pair $(c \cdot I_n, 0)$ is element of the set of admissible stresses for all $c \in \mathbb{R}$. The scalar product $\sigma : A$ (and the supremum, too) is finite only if $\operatorname{tr} A = 0$ for all matrices A. Then, $\operatorname{dev} A = A$.

The other admissible pair considered is $(\lambda A, \alpha)$ with $\lambda = \frac{\sigma_y(1+H\alpha)}{|A|}$. Here, the scalar product is

$$\begin{aligned} \sigma : A + \alpha B &= \lambda A : A + \alpha B = \lambda |A|^2 + \alpha B \\ &= \sigma_y(1+H\alpha)|A| + \alpha B \\ &= \sigma_y|A| + \alpha(\sigma_y H|A| + B) \end{aligned}$$

Since $(\sigma : A + \alpha B) \leq \varphi^*(A,B)$, the expression $(\sigma_y H|A| + B)$ must be smaller or equal to zero in order to obtain a finite supremum.

The dual functional of the dissipational functional φ reads as:

$$\varphi^*(A,B) = \begin{cases} \sigma_y|A| & \text{if } \operatorname{tr}A = 0 \wedge (\sigma_y H|A| + B) \leq 0, \\ \infty & \text{if } \operatorname{tr}A \neq 0 \vee (\sigma_y H|A| + B) > 0, \end{cases}$$

with the two arguments $A = \frac{p-p_0}{\Delta t}$ and $B = \frac{\alpha_0 - \alpha}{\Delta t}$. The minimization of (13) with respect to α affects only the term $\int_\Omega |\alpha|^2 dx$ under the restriction $(\sigma_y H|A| + B) \leq 0$. The unique solution is $\alpha = \alpha_0 + \sigma_y H|p - p_0|$. So we obtain a simplified minimization problem:

Problem 2.5. *Find the minimizer (u,p) of*

$$\begin{aligned} f(u,p) := \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u) - p] : (\varepsilon(u) - p)dx + \frac{1}{2}\int_\Omega (\alpha_0 + \sigma_y H|p - p_0|)^2 dx \\ + \int_\Omega \sigma_y|p - p_0|dx - \int_\Omega b\,u\,dx \end{aligned} \tag{14}$$

under the constraint tr $(p - p_0) = 0$.

The restriction tr $(p - p_0) = 0$ deduces from the condition tr $A = 0$. The uniqueness of the minimizer follows from the properties of the dual functional, see [6].

3 Algorithm

Time and space discretizations are needed to describe the mathematical model numerically. For the numerical tests only equidistant time intervals will be used. The notation is the same as for the time discretized problems of Section 2: For given variables (with index 0) of an initial time step t_0, the upgrades of the variables at the time step $t_1 = t + \Delta t$ have to be determined. The basic idea for solving the quasi-static problem is using a uniform time discretization and iterate in each time step until the minimizers, i.e., the displacement u and the plastic part of the strain p, are determined. Then these values and the separately calculated α are used as the reference values with index 0 for the next time step t_2.

If a function is quadratic, then the minimum can be computed easily, e.g. by Newton's method. Unfortunately, f in (14) is not. The matrix $\mathbb{C}$ is symmetric and positive definite, thus $\mathbb{C}[\varepsilon(u) - p] : (\varepsilon(u) - p)$ behaves quadratically in $(\varepsilon(u) - p)$. The second term is quadratic in p, since p_0 (the result of the previous time step) and α_0 are considered as constants. The last term behaves linearly in u, so it adds to the right hand side of the corresponding system of equations, see (19). The only term not behaving quadratically is the third one containing an norm the sharp bend of which may cause trouble.
The term $|p|$ is regularized by smoothing the norm function as follows:

$$|p|_\epsilon := \begin{cases} |p| & \text{if } |p| \geq \epsilon, \\ \frac{1}{2\epsilon}|p|^2 + \frac{\epsilon}{2} & \text{if } |p| < \epsilon. \end{cases} \tag{15}$$

For small ϵ, the function $f(u,p)$ is very similar to the original one, but its properties change enormously. Therefore, it will be referred to by the new symbol $\bar{f}$.

Another simplification is defining the change of p by $\tilde{p} = p - p_0$, and using it as an argument of the objective instead of p:

$$\begin{aligned} \bar{f}(u,\tilde{p}) := \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u) - \tilde{p} - p_0] : (\varepsilon(u) - \tilde{p} - p_0)\, dx - \int_\Omega b\, u\, dx \\ + \frac{1}{2}\int_\Omega \alpha_0^2\, dx + \frac{1}{2}\int_\Omega \sigma_y^2\, H^2\, |\tilde{p}|^2\, dx + \int_\Omega \sigma_y\, (1 + \alpha_0\, H)|\tilde{p}|_\epsilon\, dx. \end{aligned} \tag{16}$$

Now the spatial discretization is carried out by the standard finite element method using quadratic tetrahedral finite elements. For reasons of better readability and coherence, the name of the vector denoting the discretized displacement u is again u. The same is valid for $\tilde{p}$, p_0, but furthermore the symmetric matrices are transformed to vectors, e.g. in 2D

$$\begin{pmatrix} \tilde{p}_{11} & \tilde{p}_{12} \\ \tilde{p}_{12} & \tilde{p}_{22} \end{pmatrix} \Longrightarrow \begin{pmatrix} \tilde{p}_{11} \\ \tilde{p}_{22} \\ \tilde{p}_{12} \end{pmatrix},$$

such that the objective and other equations can be written in a matrix and vector notation. Now, the objective is equivalent to

$$\frac{1}{2}(Bu - \tilde{p})^T \mathbb{C}(Bu - \tilde{p}) + \frac{1}{2}\tilde{p}^T \mathbb{H}(|\tilde{p}|_\epsilon)\tilde{p} + (-B^T \mathbb{C} p_0 - b)^T u \tag{17}$$

under the constraint $\operatorname{tr}\tilde{p} = 0$. Here, Bu denotes the discretized strain $\varepsilon(u)$. $\mathbb{H}$ is the Hessian of the discretized objective with respect to $\tilde{p}$, it depends on $|\tilde{p}|_\epsilon$ and is computed as

$$\mathbb{H} = \left(\sigma_y^2 H^2 + \frac{2\sigma_y(1+\alpha_0 H)}{|\tilde{p}|_\epsilon}\right)\mathbb{Q}$$

where $\mathbb{Q}$ is the result of regarding $\tilde{p}$ as a vector and defined by

$$\text{2D: } \mathbb{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \text{3D: } \mathbb{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Hence the matrix norm is defined by $|p| = \left(p^T \mathbb{Q} p\right)^{\frac{1}{2}}$.

In order to gain a linear system of equations, the Hessian is computed in every iteration step using the current $\tilde{p}$, but apart from this the dependence on $|\tilde{p}|_\epsilon$ will be neglected. This is not an exact method to determine the change of the strain, but the error will be corrected later on as $\tilde{p}$ will be computed separately.

Since the constraint $\operatorname{tr}\tilde{p} = 0$ is linear, i.e., in 2D: $\tilde{p}_{22} = -\tilde{p}_{11}$, in 3D: $\tilde{p}_{33} = -\tilde{p}_{11} - \tilde{p}_{22}$, it is equivalent to project the problem with the matrix P onto a hyperplane, where the constraint is satisfied exactly: $\tilde{p} = P\bar{p}$ with

$$\text{2D: } P = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{3D: } P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In matrix notation the minimization problem (17) with the new variable $\bar{p}$ now reads as

$$\frac{1}{2}\begin{pmatrix} u \\ \bar{p} \end{pmatrix}^T \begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C}P \\ -P^T\mathbb{C}B & P^T(\mathbb{C}+\mathbb{H})P \end{pmatrix}\begin{pmatrix} u \\ \bar{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ P^T\mathbb{C}p_0 \end{pmatrix}^T \begin{pmatrix} u \\ \bar{p} \end{pmatrix} \longrightarrow \text{min!} \tag{18}$$

The above matrix is positive definite, thus the minimizer $(u, P\bar{p})$ has to fulfill the necessary condition of the derivative being equal to zero:

$$\begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C}P \\ -P^T\mathbb{C}B & P^T(\mathbb{C}+\mathbb{H})P \end{pmatrix}\begin{pmatrix} u \\ \bar{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ P^T\mathbb{C}p_0 \end{pmatrix} = 0. \tag{19}$$

Extracting $\bar{p}$ from the second line in (19) and inserting it into the first one yields the Schur-Complement system in u:

$$\begin{aligned} B^T(\mathbb{C} - \mathbb{C}P(P^T(\mathbb{C}+\mathbb{H})P)^{-1}P^T\mathbb{C})Bu &= \\ -b - B^T(\mathbb{C} + \mathbb{C}P(P^T(\mathbb{C}+\mathbb{H})P)^{-1}P^T\mathbb{C})\, p_0. \end{aligned} \tag{20}$$

This linear system is solved by a multigrid preconditioned conjugate gradient method, see [11]. From the numerical tests we have seen that it is not necessary to use the multigrid preconditioner arising from the plasticity problem, the preconditioner for the related problem of elasticity is sufficient and much faster. Furthermore, the nested iteration approach was used, which means that the starting values for the coarse grid correction are the restrictions of the fine grid functions.

The stopping criterion for the algorithm in each time step is

$$|\bar{f}(u^{n+1}, p^{n+1}) - \bar{f}(u^n, p^n)| \leq \varepsilon |\bar{f}(u^0, p^0)|$$

The minimization in $\tilde{p}$ can be done locally for each element, as no connections over several elements (e.g. derivatives) occur. For minimizing the function $\bar{f}$ in $\tilde{p}$, all the terms depending only on u become redundant. The remaining function, called F, becomes

$$F(\tilde{p}) = \frac{1}{2}\tilde{p}^T \mathbb{C}\tilde{p} + p_0^T \mathbb{C}\tilde{p} - \tilde{p}^T \mathbb{C}\varepsilon(u) + \frac{1}{2}\sigma_y^2 H^2 |\tilde{p}|^2 + \sigma_y(1 + \alpha_0 H)|\tilde{p}|_\epsilon. \tag{21}$$

Then, $\tilde{p}$ is determined by a modified Newton's Method, where the constraint is considered. The local Newton system to determine the search direction $\Delta\bar{p}$ writes as

$$P^T F''(\tilde{p}) P \Delta\bar{p} = -P^T F'(\tilde{p}). \tag{22}$$

4 Numerical results

The algorithm was implemented in NGSolve - the finite element solver extension pack of the mesh generation tool Netgen[1] developed in our group [15]. As finite element basis functions on the triangular, resp. tetrahedral elements we chose piecewise quadratic functions for u and piecewise constant functions for p. Furthermore, the full multigrid method was used, i.e., we started with a coarse grid, solved the problem, refined the grid, solved the problem on the finer grid et cetera.

The testing geometry is the half of a three-dimensional ring, see left figure 2 for a 2D-sketch. The problem is symmetric, so considering only the upper quarter,

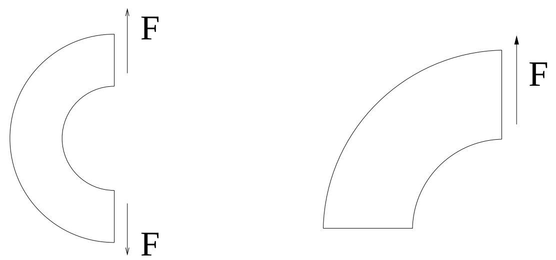

Figure 2: Halfring and reduced problem

see right figure 2, under symmetry boundary conditions is sufficient. The material

[1]Download at http://www.hpfem.jku.at/netgen

constants are chosen as $E = 1$, $\nu = 0.2$, $H = 0.01$, $\sigma_y = 1$ and the force working on the right edge is $F = 0.25$ (to fulfill the safe-load assumption). The inner quarter radius is 1, the outer radius is 2, and the thickness is 1.

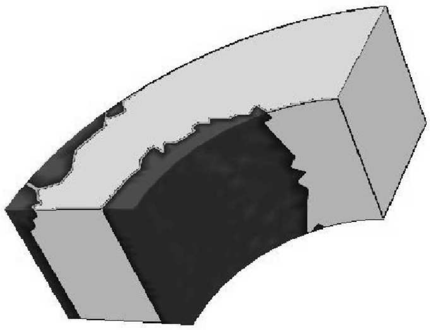

Figure 3: Plasticity domain (darkgrey)

Figure 3 shows the darkgrey domains where the material has plastified, the light-grey area is still elastic. Figure 4 shows the linear complexity of the algorithm.

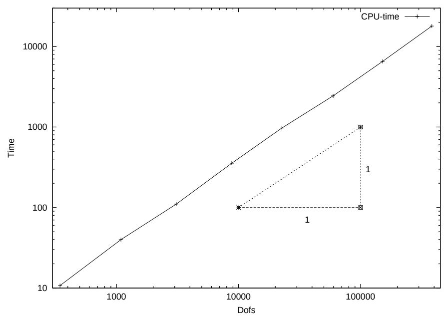

Figure 4: Linear complexity of algorithm (Dofs versus time)

5 Conclusions

In this paper the theory of elasto-plasticity has been combined with the nested iteration approach and a multigrid preconditioned conjugate gradient solver. A quasi-static algorithm solving the case of isotropic hardening in 3D has been designed, implemented and tested. The numerical results demonstrate the fast algorithm performance with linear complexity.

Our future work will concentrate on implementing the algorithm for other hardening laws and adaptive refinement techniques, and doing numerical analysis to prove the convergence observed in the numerical example.

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