FOUNTAINS, HISTOGRAMS, AND q–IDENTITIES

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ABSTRACT. We solve the recursion $S_n = S_{n-1} - q^n S_{n-p}$, both, explicitly, and in the limit for $n \to \infty$, proving in this way a formula due to Merlini and Sprugnoli. It is also discussed how computer algebra could be applied.

1. GENERALIZED SCHUR POLYNOMIALS

In the paper [5], a short and direct proof of the following fact would be required [4]: For fixed $p \geq 1$, define polynomials as follows:

$$E_n = E_{n-1} - q^n E_{n-p}, \quad n \geq p, \qquad E_0 = \cdots = E_{p-1} = 1,$$

$$D_n = D_{n-1} - q^n D_{n-p}, \quad n \geq p, \qquad D_i = 1 - \sum_{j=1}^{i} q^j, \; i = 0, \ldots, p-1.$$

Then

$$E_\infty := \lim_{n \to \infty} E_n = \sum_{k \geq 0} \frac{(-1)^k q^{p\binom{k+1}{2}}}{(1-q)\ldots(1-q^k)},$$

$$D_\infty := \lim_{n \to \infty} D_n = \sum_{k \geq 0} \frac{(-1)^k q^{k+p\binom{k}{2}}}{(1-q)\ldots(1-q^k)}.$$

We will not only achieve that but actually derive *explicit* expressions for these polynomials!

Let us study the generic recursion

$$S_n = S_{n-1} + tq^{n-p} S_{n-p},$$

with unspecified initial values $S_0, \ldots, S_{p-1}$. For $p = 2$, these polynomials were studied by Andrews (and others) in the context of *Schur polynomials*, see [2].

We will use standard notation from q–calculus, see [1]:

$$(x)_n = (1-x)(1-xq)\ldots(1-xq^{n-1}), \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k(q)_{n-k}}.$$

It will be convenient to define $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $n < 0$ or $k > n$.

Now we will proceed as in [1] and consider noncommutative variables x, η, such that $x\eta = q\eta x$; all other variables commute.

Lemma 1.

$$\left(x + x^p\eta\right)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{p\binom{n}{2} - pnk + p\binom{k+1}{2}} x^{k+p(n-k)}\eta^{n-k}.$$

Proof. We write

$$\left(x + x^p\eta\right)^n = \sum_{k=0}^{n} a_{n,k} x^{k+p(n-k)}\eta^{n-k},$$

and $\left(x + x^p\eta\right)^{n+1} = \left(x + x^p\eta\right)^n \left(x + x^p\eta\right)$ resp. as $\left(x + x^p\eta\right)^{n+1} = \left(x + x^p\eta\right)\left(x + x^p\eta\right)^n$, compare coefficients, and get the recursions

$$a_{n+1,k} = a_{n,k-1} + a_{n,k}q^{k+p(n-k)},$$
$$a_{n+1,k} = a_{n,k-1}q^{n+1-k} + a_{n,k}q^{p(n-k)}.$$

From this we derive, taking differences,

$$a_{n,k} = \frac{1 - q^{n+1-k}}{1 - q^k} q^{-p(n-k)} a_{n,k-1}.$$

The result follows from iteration by noting that $a_{n,0} = q^{p\binom{n}{2}}$. $\qquad\square$

Of course we also have

$$\left(x + tx^p\eta\right)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{p\binom{n}{2} - pnk + p\binom{k+1}{2}} x^{k+p(n-k)}t^{n-k}\eta^{n-k}.$$

Now we derive the generating function for

$$F(x) = \sum_{n \geq 0} S_n x^n;$$

the following procedure is inspired by [2]. Note that we can alternatively view η as an operator, defined by $\eta f(x) = f(qx)$. Cigler worked also much with this technique [3]. We find

$$\sum_{n \geq p} S_n x^n = \sum_{n \geq p} S_{n-1} x^n + \sum_{n \geq p} tq^{n-p} S_{n-p} x^n = x \sum_{n \geq p-1} S_n x^n + tx^p \sum_{n \geq 0} \eta S_n x^n$$

or

$$F(x) - \sum_{n < p} S_n x^n = xF(x) - x \sum_{n < p-1} S_n x^n + tx^p\eta F(x),$$

and

$$F(x) = \frac{1}{1 - x - tx^p\eta}\left(\sum_{i < p} S_i x^i - \sum_{i < p-1} S_i x^{i+1}\right).$$

Now we can apply our lemma and write

$$F(x) = \sum_{n \geq 0} \left(x + tx^p\eta\right)^n \left(\sum_{i < p} S_i x^i - \sum_{i < p-1} S_i x^{i+1}\right)$$

$$= \sum_{n \geq 0} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{p\binom{n}{2} - pnk + p\binom{k+1}{2}} x^{k+p(n-k)} t^{n-k} \eta^{n-k} \left(\sum_{i < p} S_i x^i - \sum_{i < p-1} S_i x^{i+1}\right)$$

$$= \sum_{n \geq 0} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{p\binom{n}{2} - pnk + p\binom{k+1}{2}} x^{k+p(n-k)} t^{n-k} \left(\sum_{i < p} S_i q^{i(n-k)} x^i - \sum_{i < p-1} S_i q^{(i+1)(n-k)} x^{i+1}\right)$$

$$= \sum_{n \geq 0} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{p\binom{k}{2}} x^{n-k+pk} t^k \left(\sum_{i<p} S_i q^{ik} x^i - \sum_{i<p-1} S_i q^{(i+1)k} x^{i+1} \right)$$

$$= \sum_{k,n \geq 0} \begin{bmatrix} n+k \\ k \end{bmatrix} q^{p\binom{k}{2}} x^{n+pk} t^k \left(\sum_{i<p} S_i q^{ik} x^i - \sum_{i<p-1} S_i q^{(i+1)k} x^{i+1} \right)$$

$$= \sum_{k \geq 0} q^{p\binom{k}{2}} x^{pk} t^k \frac{1}{(x)_{k+1}} \left(\sum_{i<p} S_i q^{ik} x^i - \sum_{i<p-1} S_i q^{(i+1)k} x^{i+1} \right).$$

From this we find an explicit formula for S_n (the quantity S_{-1} has to be interpreted as 0):

$$S_n = \sum_{0 \leq i < p} (S_i - S_{i-1}) \sum_{k \geq 0} \begin{bmatrix} n - (p-1)k - i \\ k \end{bmatrix} q^{p\binom{k}{2}+ik} t^k.$$

Now we specialize this to our instance. Here, $t = -q^p$, and thus

$$S_n = \sum_{0 \leq i < p} (S_i - S_{i-1}) \sum_{k \geq 0} \begin{bmatrix} n - (p-1)k - i \\ k \end{bmatrix} q^{p\binom{k+1}{2}+ik} (-1)^k.$$

Therefore

$$E_n = \sum_{k \geq 0} \begin{bmatrix} n - (p-1)k \\ k \end{bmatrix} q^{p\binom{k+1}{2}} (-1)^k.$$

From this, the limit of E_n is immediate. For D_n we eventually get the following form

$$D_n = \sum_{k \geq 0} \begin{bmatrix} n - (p-1)(k-1) \\ k \end{bmatrix} q^{k+p\binom{k}{2}} (-1)^k,$$

from which the formula for D_∞ is immediate. To prove it, we need a simple lemma whose proof is just a routine calculation.

Lemma 2.

$$\begin{bmatrix} m - i \\ k \end{bmatrix} q^{i(k+1)} = g(i) - g(i-1) \qquad where \qquad g(i) = - \begin{bmatrix} m - i \\ k+1 \end{bmatrix} q^{(i+1)(k+1)}. \quad \square$$

Now we can plug into the general formula above and compute

$$D_n = E_n - \sum_{i=1}^{p-1} \sum_{k \geq 0} \begin{bmatrix} n - (p-1)k - i \\ k \end{bmatrix} q^{p\binom{k+1}{2}+i(k+1)} (-1)^k$$

$$= E_n - \sum_{k \geq 0} (-1)^k q^{p\binom{k+1}{2}} \sum_{i=1}^{p-1} \begin{bmatrix} n - (p-1)k - i \\ k \end{bmatrix} q^{i(k+1)}$$

$$= E_n - \sum_{k \geq 0} (-1)^k q^{p\binom{k+1}{2}} \left\{ q^{k+1} \begin{bmatrix} n - (p-1)k \\ k+1 \end{bmatrix} - q^{p(k+1)} \begin{bmatrix} n - (p-1)(k+1) \\ k+1 \end{bmatrix} \right\}$$

$$= 1 - \sum_{k \geq 0} (-1)^k q^{p\binom{k+1}{2}} q^{k+1} \begin{bmatrix} n - (p-1)k \\ k+1 \end{bmatrix},$$

which is the announced formula after a simple change of variable. Note that in the penultimate step the telescoping property of the lemma has been used.

2. Computer algebra proofs

The polynomial families (E_n) and (D_n) give rise to the following study with respect to possible computer proofs. Let us take as input our sum representations of E_n and D_n:

$$
\begin{aligned}
E_n &= \sum_{k \geq 0} \begin{bmatrix} n - (p-1)k \\ k \end{bmatrix} q^{p\binom{k+1}{2}} (-1)^k, \\
D_n &= \sum_{k \geq 0} \begin{bmatrix} n - (p-1)(k-1) \\ k \end{bmatrix} q^{k+p\binom{k}{2}} (-1)^k.
\end{aligned}
\tag{1}
$$

Then, if p is chosen as a specific positive integer, Riese's package qZeil [6] returns the recurrences $S_n = S_{n-1} - q^n S_{n-p}$ $(n \geq p)$ together with a certificate function Cert for independent verification. Despite the fact that for general "generic" integer parameter p there is no algorithm available, a general pattern can be easily guessed from running the algorithm for $p = 1$, $p = 2$, and $p = 3$, say.

For example, let $F(n, k)$ be the kth summand in our sum representation (1) of E_n, then the recurrence for E_n can be refined to the following statement.

Theorem 1. *For $n \geq p$ and $\delta_k f(n,k) = f(n,k) - f(n, k-1)$, we have*

$$
F(n, k) - F(n-1, k) + q^n F(n-p, k) = \delta_k \mathsf{Cert}(n, k) F(n, k),
\tag{2}
$$

where

$$
\mathsf{Cert}(n, k) = q^n \frac{(q^{n-p(k+1)+1})_p}{(q^{n-(p-1)(k+1)})_p}.
$$

Proof. After dividing both sides of (2) by $F(n, k)$ the proof reduces to checking equality of rational functions. Namely, note that

$$
\begin{aligned}
\frac{F(n-1, k)}{F(n, k)} &= \frac{1 - q^{n-pk}}{1 - q^{n-(p-1)k}}, \\
\frac{F(n, k-1)}{F(n, k)} &= -\frac{q^{pk}}{1 - q^k} \frac{(q^{n-pk+1})_p}{(q^{n-(p-1)k+1})_{p-1}},
\end{aligned}
$$

and

$$
\frac{F(n-p, k)}{F(n, k)} = q^{-n} \mathsf{Cert}(n, k). \quad \square
$$

Analogously, there is a refined version of the recurrence for D_n. The certificate in this case is

$$
\mathsf{Cert}(n, k) = q^n \frac{(q^{n-pk})_p}{(q^{n-(p-1)k})_p}.
$$

Summarizing, with the sum representation for E_n and D_n in hand, the corresponding recurrences follow immediately by summing both sides of the computer recurrences (2) over all $k \geq 0$.

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