Growth Fronts of First-Order Hamilton-Jacobi Equation

Martin Burger *

Industrial Mathematics Institute, University Linz,
Altenberger Strasse 69, A-4040 Linz, Austria.
e-mail: burger@indmath.uni-linz.ac.at

Abstract

The aim of this paper is to investigate the propagation of fronts for a class of first-order Hamilton-Jacobi equations, where certain properties of the Hamiltonian imply that the level set $\{u(.,t) \leq 0\}$ of the solution u is growing with respect to time. Besides monotonicity of this level set, we show that the number of its connected components is nonincreasing with respect to time and derive lower and upper bounds for these sets.

By using lower bound gradient estimates for the solutions of first-order Hamilton-Jacobi equations, recently obtained by Ley [17], we prove a regularity result for the growth front, which holds for almost all times t. Finally, we derive an estimate of the Hausdorff measure of the front interpreted as an evolving hypersurface.
Keywords: Hamilton-Jacobi Equations, Level Set Methods, Growth, Front Propagation, Viscosity Solutions.
Subject Classification (MSC 2000): 70H20, 35R35, 35L45, 53C99

1 Introduction

The theory of first- and second-order Hamilton-Jacobi equations has been developed during the last decades, since the concept of *viscosity solutions* was introduced (cf. e.g. [19, 2]). The first motivation to study this type of solutions arises from the dynamic programming principle in optimal control, which states that the value function solves a Hamilton-Jacobi equation, whose Hamiltonian is related to the Lagrangian of the control problem via the Legendre transform.

In the present paper we shall be concerned in particular with another class of applications for Hamilton-Jacobi equations, namely front propagation problems, modeled by the level set method. A well-known example in this class is the *mean curvature motion*, whose mathematical analysis via the level set approach has led to important results (cf. [3, 8, 11, 12, 13, 14]). Our analysis is concerned with first-order equations of the form

$$\frac{\partial u}{\partial t} + H(x,t,\nabla u) = 0 \qquad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.1}$$

*Supported by the Austrian Science Foundation under project SFB F 13/08

where ∇u denotes the gradient of $u(.,t)$ with respect to the space variable, and in particular with the evolution of their level sets

$$\Gamma(t) := \{ \ x \in \mathbb{R}^N \ \mid u(x,t) = 0 \ \}, \tag{1.2}$$

respectively,

$$\Omega(t) := \{ \ x \in \mathbb{R}^N \ \mid u(x,t) \leq 0 \ \}. \tag{1.3}$$

Under standard assumptions on the Hamiltonian (given below), equation (1.1) with initial value

$$u(.,t) = u_0 \in C_b^{0,1}(\mathbb{R}^N) \tag{1.4}$$

has a unique viscosity solution $u \in C(\mathbb{R}^+; C_b^{0,1}(\mathbb{R}^N))$. Moreover, if the Hamiltonian H satisfies

$$H(x,t,p) \text{ is convex in the } p-\text{variable}, \qquad \forall\ (x,t) \in \mathbb{R}^N \times \mathbb{R}_0^+ \tag{1.5}$$

and is homogeneous of degree one, i.e.,

$$H(x,t,\lambda p) = \lambda H(x,t,p), \qquad \forall\ (x,t,p) \in \mathbb{R}^N \times \mathbb{R}_0^+ \times \mathbb{R}^N, \forall\ \lambda \in \mathbb{R}^+, \tag{1.6}$$

then due to the results by Ley [17], the norm of the generalized gradient ∇u (in the sense of gradients for Lipschitz continuous functions) is uniformly bounded below by a positive constant in a neighborhood of the zero level set $\Gamma(t)$ for each finite time interval. This lower bound estimate implies that *fattening* cannot occur, i.e, the set $\Gamma(t)$ does not create any interior in finite time, i.e., $\Gamma(t)$ has zero Lebesgue measure in $\mathbb{R}^N$ provided the initial curve

$$\Gamma(0) = \Gamma_0 = \{ \ x \in \mathbb{R}^N \ \mid u_0(x) \leq 0 \ \} \tag{1.7}$$

has zero Lebesgue measure. To the author's knowledge, this is the first result on non-fattening for a general class of equations, while in the previous literature only examples of fattening or non-fattening, such as for the mean curvature equation (cf. e.g. [11, 5]), were known.

Moreover, the lower bound gradient estimate can be used to show that (for semiconcave solutions, cf. [17] for details) the Clarke subgradient of does not contain 0 almost everywhere, which is the basis for a nonsmooth inversion theorem. We note that if the solution was smooth (i.e., at least in the class C^1) and the norm of its gradient is bounded away from zero, then the implicit function theorem would directly imply that $\Gamma(t)$ is a smooth curve. However, solutions of (1.1) are in general only of Lipschitz type, and the analogue of the implicit function theorem for Lipschitz-continuous functions gives results only almost everywhere, which prevents to conclude any regularity property for the front by this technique. Ley [17] could at least obtain partial regularity for Γ^α for almost every α in the neighborhood of 0 and recover a positive result by Barles et al. [3] on the so-called *non-empty interior difficulty*, which means the question if the front $\Gamma(t)$ has zero Lebesgue measure.

In this paper we improve the results of Ley [17] for a specific class of Hamiltonians, namely those satisfying in addition

$$A_1 + A_2\,|p| \geq H(x,t,p) \geq a_1 + a_2\,|p|\,, \qquad \forall\ (x,t,p) \in \mathbb{R}^N \times \mathbb{R}_0^+ \times \mathbb{R}^N, \tag{1.8}$$

for given nonnegative constants a_i, A_i, $i = 1,2$ with $A_1 + A_2 \geq a_1 + a_2 > 0$. Under this assumption on the Hamiltonian, it turns out that sets $\Omega(t)$ are increasing with respect to time, i.e., the front propagation describes a growth model. We will show that the number of connected components of $\Omega(t)$ and of $\Gamma(t)$ are nonincreasing during the evolution and establish a comparison with the level sets obtained from (1.1), with the Hamiltonians

$$H^+(x,t,p) = A_1 + A_2\,|p|\,, \qquad H^-(x,t,p) = a_1 + a_2\,|p|\,. \tag{1.9}$$

Finally, we establish a regularity result for the growth front $\Gamma(t)$ yielding local Lipschitz regularity almost everywhere, which holds for almost all $t \in \mathbb{R}^+$, and therefore improves the results by Ley [17], who showed local Lipschitz regularity of the level sets

$$\Gamma^\alpha(t) = \{\ x \in \mathbb{R}^N \ \mid u(x,t) = \alpha\ \}, \tag{1.10}$$

for almost all $\alpha \in (-\alpha_0, \alpha_0)$, with some $\alpha_0 > 0$, from which one cannot deduce any result on the growth front $\Gamma(t)$ itself. The basic idea of our proof is to characterize the growth front equivalently as

$$\Gamma(t) = \{\ x \in \mathbb{R}^N \ \mid \tau(x) = t\ \}, \tag{1.11}$$

where $\tau(x)$ is the *arrival time* of the front defined by

$$\tau(x) := \inf\{\ t \in \mathbb{R}^+ \ \mid u(x,t) = 0\ \} = \inf\{\ t \in \mathbb{R}^+ \ \mid \phi(x,t) \le 0\ \}. \tag{1.12}$$

Using the lower bound gradient estimate for u we can show τ is well-defined and locally Lipschitz-continuous, which serves as a basis for our regularity result, since the evolution of the growth front is determined by the level sets of τ.

The remainder of the paper is organized as follows: in Section 1.1 we introduce some notations and basic assumptions needed below. In Section 1.2 we present the main class of applications we have in mind, namely level set methods for normal growth of multiple objects, and discuss the validity of our assumptions for the specific type of Hamiltonian appearing in such models. Section 2 reviews some concepts of generalized solutions of Hamilton-Jacobi equations and some results on their existence, uniqueness, and regularity. These results are applied to the solutions of (1.1) in Section 3, and subsequently to the propagating front in Section 4, which includes the main results of this paper.

1.1 Notation and Assumptions

In the following we introduce the basic notations and assumptions used throughout the paper. We start with some definitions and then formulate assumptions on the initial value u_0 and on the Hamiltonian H needed in the remainder of the paper.

Unless further noticed, we always work in the spatial domain $\mathbb{R}^N$ ($N \in \mathbb{N}$ being arbitrary), the Lebesgue measure of subsets will be denoted by λ^N, and the d-dimensional Hausdorff-measure by $\mathcal{H}^d$ (with $d = N-1$ being of particular importance), referring to [15, 20] definitions and properties of these measures. By the letter Ω (combined with indices) we will in general denote sets of positive Lebesgue measure in $\mathbb{R}^N$, while the letter Γ is used for sets (hyper-surfaces) with zero Lebesgue-measure λ^N and possibly positive Hausdorff-measure $\mathcal{H}^{N-1}$. Moreover, we use the notation

$$d_K(x) = \inf_{y\in K} |x-y|$$

for the distance of a point x from a compact set K, and define the signed distance function for the boundary of a compact set via

$$\sigma_{\partial K}(x) = \begin{cases} d_{\partial K}(x) & \text{if } x \notin K \\ -d_{\partial K}(x) & \text{if } x \in K \end{cases}. \tag{1.13}$$

For open or closed sets $K \subset \mathbb{R}^d$ we will use the standard notions of continuity and differentiability, denoting the total derivative of order j of a function f by $D^{(j)}f$, and partial derivatives with respect to a variable x by the standard symbol $\frac{\partial f}{\partial x}$, if $x \in \mathbb{R}^N$ is the spatial variable also by ∇f. The space of bounded continuous functions on K will be denoted by $C(K)$, if K is compact and $C_b(K)$ else. In general, the space of k-times continuously differentiable functions on K shall be denoted by $C^k(K)$ (respectively by $C_b^k(K)$ for non-compact K, $k \in \mathbb{N}$), with norm defined by

$$\|f\|_{C^k_{(b)}(K)} = \max_{0\leq j\leq k} \sup_{x\in K} \left|D^{(j)}f(x)\right|. \tag{1.14}$$

We also introduce the Hölder spaces $C^{k,\beta}(K)$ for $k \in \mathbb{N}\cup\{0\}$ and $\beta \in (0,1]$ with norm

$$\|f\|_{C^{k,\beta}_{(b)}(K)} = \max\left\{\|f\|_{C^k(K)}, \sup_{x,y\in K} \frac{\left|D^{(k)}u(x) - D^{(k)}u(y)\right|}{|x-y|^\beta}\right\}. \tag{1.15}$$

For functions f mapping into a general Banach space X instead of $\mathbb{R}^d$ for some $d \in \mathbb{N}$ we will use the specific notations $C(K;X)$, $C^k(K;X)$ and $C^{k,\beta}(K;X)$. The norms in these spaces are the same as above with the Euclidean norm $|.|$ in $\mathbb{R}^d$ replaced by the norm in X (cf. [26] for details). Finally, we use the notation $L^\infty(K)$ for the space of bounded measurable functions on K, equipped with the norm

$$\|u\|_{L^\infty(K)} = \text{ess} \sup_{x\in K} |u(x)|,$$

and the notation $W^{1,\infty}(K)$ for the space of bounded measurable functions, whose distributional derivative is again bounded and measurable. The norm in this space is given by

$$\|u\|_{W^{1,\infty}(K)} = \max\{\|u\|_{L^\infty(K)}, \|Du\|_{L^\infty(K)}\}.$$

We assume in the following without further notice that the initial value satisfies $u_0 \in C_b^{0,1}(\mathbb{R}^N)$. For the Hamiltonian $H \in C(\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^N)$ we will use the standard regularity assumptions

$$\left|\frac{\partial H}{\partial x}(x,t,p)\right| \leq C_1(\beta + |p|), \tag{1.16}$$

with nonnegative constants C_1, β, and

$$\left|\frac{\partial H}{\partial p}(x,t,p)\right| \leq C_2\,|x| + C_3, \tag{1.17}$$

with nonnegative C_2, C_3, for almost all $(x,t,p) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^N$. If needed, we shall also use the assumptions (1.5), (1.6) and (1.8).

1.2 Level Set Methods for Normal Growth

The general results on the class of Hamilton-Jacobi equations under investigation can be applied directly to an important type of equations frequently used in the modeling of front-propagation problems with prescribed normal speed, namely *level set equations* of the form

$$\frac{\partial u}{\partial t} + v\,|\nabla u| = 0 \qquad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \tag{1.18}$$

where v is an extension of the normal velocity of the evolving front $\Gamma(t)$ to $\mathbb{R}^N \times (0,T)$. Level set methods have been used with great success as a computational tool for propagation of fronts since they were introduced by Osher and Sethian [22], with a variety of practical applications such as flame propagation, fluid dynamics, materials science, or image processing (cf. [21, 23] for an overview).

While the analysis of the level set equation (1.18) itself is well-established, there are still unsatisfactory gaps in the analysis of the propagating front $\Gamma(t)$ defined by (1.2) and its geometric properties. The existence and uniqueness of solutions of (1.18) follows from standard results on Hamilton-Jacobi equations provided v satisfies some standard smoothness assumptions. The properties of the front $\Gamma(t)$ could not be analyzed under such general conditions so far.

In order to satisfy the above assumptions on the Hamiltonian, we assume that v is bounded Lipschitz in $\mathbb{R}^N$ and that there exist constants $A_2 > a_2 > 0$ such that (1.19) holds. Under these assumptions on v, one easily verifies that the Hamiltonian $H(x,t,p) := v(x,t)\,|p|$ satisfies the assumptions (1.5), (1.6),(1.8), (1.16), and (1.17).

The results of this paper are directly applicable to level set methods, if the velocity v is Lipschitz continuous with respect to x uniformly in t and, if

$$A_2 \geq v(x,t) \geq a_2, \qquad \forall\ (x,t) \in \mathbb{R}^N \times R^+. \tag{1.19}$$

Under these assumptions on v, one easily verifies that the Hamiltonian $H(x,t,p) := v(x,t)\,|p|$ satisfies the assumption (1.8). Due to the nonnegativity of v, we obtain that the convexity assumption (1.5) is satisfied, and (1.8) holds with $a_1 = A_1 = 0$. The positive homogeneity (1.6) always holds for the special Hamiltonian in (1.18). Finally, the smoothness conditions (1.16) with $\beta = 0$ and (1.17) follow from the Lipschitz continuity and boundedness of v. We want to mention that the upper bound on v is reasonable for most practical applications, since v represents the normal speed of a propagating front (respectively its extension to $\mathbb{R}^N$), which usually remains finite. The lower bound means that the front will grow in a monotone way during the evolution, which is a strong restriction. However, this class of growth models still includes important applications such as the growth of polymer crystals (cf. [6, 7, 16]), where the velocity is determined by $v(x,t) = G(\theta(x,t))$ with θ being the temperature and G a given positve material function (cf. [7] for details). So far, this growth model, coupled to a diffusion equation for temperature, has been analyzed only in the case of a single crystal (cf. [16]) using strong regularity or in the spatially one-dimensional case for multiple crystals (cf. [6]). Other interesting application are photolitography development (cf. [24, 25], where the speed function depends on the spatial variable only. Besides these specific applications, we hope that the results of this paper also stimulate the analysis of the front under different conditions on the velocity.

2 Generalized Solutions of Hamilton-Jacobi Equations

In general, classical solutions of Hamilton-Jacobi equations need not exist even if the initial value and the Hamiltonian are smooth, so that concepts of generalized solutions are needed. The definition of such generalized solutions is not unique and the usefulness of specific notions depends on several factors such as the smoothness of the initial value or the structure of the Hamiltonian. In most parts of the paper, we restrict our attention to the most important notion of generalized solutions, namely the one of viscosity solutions (noticing that most standard notions are equivalent if the initial value is Lipschitz continuous and H satisfies (1.5), (1.16) and (1.17)); for some proofs we will also use the notion of *Barron-Jensen* solutions (see below). Viscosity solutions are defined as follows:

Definition 2.1. A function $u \in C(\mathbb{R}^N \times (0,T))$ is called *viscosity subsolution* of (1.1) if for every function $\phi \in C^1(\mathbb{R}^N \times (0,T))$, the inequality

$$\frac{\partial \phi}{\partial t}(\overline{x},\overline{t}) + H(\overline{x},\overline{t},\nabla\phi(\overline{x},\overline{t})) \leq 0 \tag{2.1}$$

holds for each local maximum $(\overline{x},\overline{t}) \in \mathbb{R}^N \times (0,T)$ of $u - \phi$.

A function $u \in C(\mathbb{R}^N \times (0,T))$ is called *viscosity supersolution* of (1.1) if for every function $\phi \in C^1(\mathbb{R}^N \times (0,T))$, the inequality

$$\frac{\partial \phi}{\partial t}(\overline{x},\overline{t}) + H(\overline{x},\overline{t},\nabla\phi(\overline{x},\overline{t})) \geq 0 \tag{2.2}$$

holds for each local minimum $(\overline{x},\overline{t}) \in \mathbb{R}^N \times (0,T)$ of $u - \phi$.

A function $u \in C(\mathbb{R}^N \times (0,T))$ is called *viscosity solution* of (1.1) if it is both a viscosity sub- and supersolution.

Sometimes we shall also use the notion *viscosity solution of* (1.1), (1.4), by which we mean a viscosity solution of (1.1) that satisfies in addition the initial condition (1.4).

The theory of continuous viscosity solutions is well-developed, we refer to [10, 19, 2] and the references therein for details and restrict our attention to some particular results needed in the subsequent analysis. The first one, concerned with the existence and uniqueness of viscosity solutions, could be obtained in fact under weaker assumptions on the Hamiltonian H as used in this paper, but for simplicity we use the above conditions in the formulation:

Theorem 2.2 (Existence and Uniqueness). *Let u_0 be a bounded uniformly continuous function and let $H \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R}^N)$ satisfy* (1.16), (1.17)*. Then there exists a unique continuous viscosity solution of* (1.1), (1.4)*, which is bounded uniformly continuous with respect to x uniformly in t.*

A second result on viscosity solutions is concerned with their regularity in the class of Lipschitz continuous functions, for which the Lipschitz continuity of the Hamiltonian is essential:

Theorem 2.3 (Regularity). *[17, Theorem 4.1] Let u_0 be Lipschitz continuous and let $H \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R}^N)$ satisfy* (1.16), (1.17)*. Then the unique continuous viscosity solution u is Lipschitz continuous with respect to x uniformly in t, and*

$$\|\nabla u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leq 2e^{\frac{5C_1T}{4}} \left(\|\nabla u_0\|^2_{L^\infty(\mathbb{R}^N)} + \frac{\beta C_1 T}{2} \right)^{\frac{1}{2}}, \qquad \forall\ t \in [0,T]. \tag{2.3}$$

Finally, we recall a standard comparison result for viscosity sub- and supersolutions (which could be deduced under weaker assumptions on H as Theorem 2.2):

Theorem 2.4 (Comparison). *Let H be as in Theorem 2.2, let v be a continuous viscosity subsolution and w a continuous viscosity supersolution of* (1.1) *with $w(x,0) \geq v(x,0)$. Then*

$$w(x,t) \geq v(x,t) \qquad \forall\ (x,t) \in \mathbb{R}^N \times [0,T]. \tag{2.4}$$

An alternative notion of generalized solutions of first-order Hamilton-Jacobi equations (with Hamiltonian convex with respect to p) are Barron-Jensen solutions, which were introduced to study semicontinuous solutions (cf. [4]). For our purpose, however, it is sufficient to restrict the definition to continuous solutions:

Definition 2.5. A function $u \in C(\mathbb{R}^N \times (0,T))$ is called *Barron-Jensen solution* of (1.1) if for every function $\phi \in C^1(\mathbb{R}^N \times (0,T))$, the equality

$$\frac{\partial \phi}{\partial t}(\overline{x},\overline{t}) + H(\overline{x},\overline{t},\nabla\phi(\overline{x},\overline{t})) = 0 \tag{2.5}$$

holds for each local minimum $(\overline{x},\overline{t}) \in \mathbb{R}^N \times (0,T)$ of $u - \phi$.

The equivalence of viscosity solutions and Barron-Jensen solutions is provided by the following result:

Theorem 2.6. *[17, Theorem 3.1] Let* (1.5) *and* (1.16) *hold, and let* $u \in C(\mathbb{R}^N \times (0,T))$. *Then* u *is a viscosity solution of* (1.1) *if and only if* u *is a Barron-Jensen solution of* (1.1).

3 Properties of the Viscosity Solution

In the following we derive some preliminary results on the viscosity solutions of the Hamilton-Jacobi equation (1.1) subject to (1.4), which we denote by u. Having a comparison of the growth front with fronts propagating with slower or faster speed in mind, we introduce the functions u_- and u_+, being the unique viscosity solutions of

$$\frac{\partial u_-}{\partial t} + (a_1 + a_2 |\nabla u_-|) = 0 \qquad \text{in } \mathbb{R}^N \times (0,T) \tag{3.1}$$

$$\frac{\partial u_+}{\partial t} + (A_1 + A_2 |\nabla u_+|) = 0 \qquad \text{in } \mathbb{R}^N \times (0,T) \tag{3.2}$$

with

$$u_-(.,0) = u_+(.,0) = u_0, \qquad \text{in} \mathbb{R}^N. \tag{3.3}$$

The zero level sets of these functions will serve as lower and upper bounds on the sets covered by the growth front obtained (1.1) in the subsequent analysis.

We start with an interpretation of u_- and u_+ as viscosity super- and subsolutions of (1.1):

Lemma 3.1. *Let the functions* u_- *and* u_+ *be the viscosity solutions defined by* (3.1),(3.3) *and* (3.2), (3.3), *respectively. Then* u_- *is a viscosity supersolution and* u_+ *is a viscosity subsolution of* (1.1).

Proof. Let $\phi \in C^1(\mathbb{R}^N \times (0,T))$ and suppose $u_- - \phi$ attains a local maximum at some point $(x,t) \in \mathbb{R}^N \times (0,T)$. Then, since u_- is a viscosity solution of (3.1), we obtain with (1.8) that

$$\frac{\partial \phi}{\partial t}(x,t) + H(x,t,\nabla\phi(x,t)) \geq \frac{\partial \phi}{\partial t}(x,t) + a_1 + a_2 |\nabla\phi(x,t)| \geq 0,$$

and hence, u_- is a viscosity supersolution of (1.1). The reasoning for u_+ is analogous, using the second estimate in (1.8). □

As a direct consequence of Lemma 3.1 and the comparison in Theorem 2.4 we obtain the following comparison result:

Corollary 3.2. *Let u be the unique viscosity solution of* (1.1), (1.4) *and let u_- and u_+ be as in Lemma 3.1. Then, for all $t \in (0,T)$, the relation*

$$u_+(x,t) \le u(x,t) \le u_-(x,t), \qquad \forall\, x \in \mathbb{R}^N \tag{3.4}$$

holds.

3.1 Lower Bound Gradient Estimates

Now we turn our attention to lower bound gradient estimates for viscosity solutions, a remarkable and unusal result in the theory of partial differential equations:

Theorem 3.3. *[17, Theorem 4.2] Let in addition to the assumptions of Theorem 2.3 the condition* (1.5) *be satisfied.*

(i) Let $x_0 \in \mathbb{R}^N$ and $r > 0$. If $|\nabla u_0| \ge \eta$ in $B(x_0,r)$ in the viscosity sense for some $\eta > 0$, then there exist some positive constants $\overline{\eta}$, γ and $t_0 \in (0,T]$ such that

$$|\nabla u| \ge e^{-\frac{\gamma t}{2}}\overline{\eta} \qquad \text{in } \mathcal{D}(x_0,r) \cap (\mathbb{R}^N \times (0,t_0)) \tag{3.5}$$

holds in the viscosity sense, where

$$\mathcal{D}(x_0,r) = \{\ (x,t) \in B(x_0,r) \times (0,T)\ |\ \le e^{(C_2+C_3+C_2|x_0|)t}(1+|x-x_0|)r+1\ \}$$

(ii) Let (1.6) *and* (1.16) *be satisfied with $\beta = 0$. If*

$$|u_0(x)| + |\nabla u_0(x)| \ge \eta, \tag{3.6}$$

for some constant $\eta \in \mathbb{R}^+$, then there exist positive constants C and γ such that

$$|u(.,t)| + e^{\gamma t}\,|\nabla u(.,t)|^2 \ge C \qquad \text{in } \mathbb{R}^N, \forall\, t \in (0,T)$$

in the viscosity sense.

This lower bound gradient estimate cannot be applied to u only, but also in a particular way to the functions u_- and u_+, which are viscosity solutions of an equation with Hamiltonian independent of the spatial variable x. Since an inspection of the proof of Theorem 3.3 shows that one may chose $\gamma = 0$, $\overline{\eta} = \eta$, and $t_0 = \min\{T, \frac{\ln(1+r)}{C_3}\}$ in this case, we obtain the following result:

Lemma 3.4. *Let $x_0 \in \mathbb{R}^N$ and $r > e^{C_3T} - 1$. Then, if the unique viscosity solution u of* (1.1), (1.4) *satisfies $|\nabla u_0| \ge \eta$ in $B(x_0,r)$ in the viscosity sense, then the functions u_- and u_+ defined as in Lemma 3.1 satisfy*

$$|Du_-| \ge \eta, \qquad |Du_+| \ge \eta \tag{3.7}$$

in the viscosity sense in $B(x_0,\rho) \times (0,T)$ for $\rho \le (1+r)e^{-C_3T} - 1$.

Finally, we note that if $|\nabla u_0| = 1$ in a sufficiently large ball, then an application of Lemma 3.4 and Theorem 2.3 (whose assumptions are satisfied by H_+ and H^- with $C_1 = 0$) shows that $|\nabla u_\pm| = 1$ in the viscosity sense in $U \times (0,T)$, for some neigbourhood U of the zero level set $\cap_{t\in(0,T)}\Gamma(t)$. This property is of particular interest in applications to level set methods, where the the signed distance function $\sigma_{\Gamma(0)}$ for the curve $\Gamma(0) = \partial\Omega(0)$ is frequently used as an initial value for u.

3.2 Decay of Solutions

In the preceding part we have obtained a lower bound gradient estimate for the unique viscosity solution of (1.1), (1.4), which we apply now to estimate the decay of the solution. If u was a classical solution and $|\nabla u| \geq \kappa$ in $U \times (s,S)$ in the viscosity sense for some $U \subset \mathbb{R}^N$ and $0 \leq s < S \leq T$, then from (1.8) we would obtain that

$$\frac{\partial u}{\partial t}(x,t) = -H(x,t,\nabla u(x,t)) \leq -(a_1 + a_2\kappa), \qquad \forall\ (x,t) \in U \times (s,S),$$

and consequently the decay of u would be at least

$$u(x,t) \leq u(x,s) - (a_1 + a_2\kappa)(t-s), \qquad \forall\ (x,t) \in U \times (s,S). \tag{3.8}$$

In the following we will generalize these inequalities to viscosity solutions, i.e., we show that a variant of the first one holds in the viscosity sense, and the second one follows from a comparison result.

Lemma 3.5. *Let $u \in C(\mathbb{R}^N \times (0,T)$ be a viscosity solution of* (1.1) *with $|\nabla u| \geq \kappa > 0$ in $U \times (s,S)$ in the viscosity sense, and let the Hamiltonian H satisfy* (1.5), (1.8), (1.16) *and* (1.17). *Then u is a viscosity subsolution of*

$$\frac{\partial w}{\partial t} + (a_1 + a_2 f) = 0, \tag{3.9}$$

in $\mathbb{R}^N \times (s,S)$ for any continuous function $f : \mathbb{R}^N \to \mathbb{R}$ satisfying $f \equiv 0$ in $\mathbb{R}^N - U$ and $0 \leq f(x) \leq \kappa$ for $x \in U$.

Proof. First of all, a direct calculation shows that $|\nabla u| \geq f$ in the viscosity sense in $\mathbb{R}^N \times (0,T)$. Let $\phi \in C^1(R^N \times (0,T))$ and let $(\overline{x},\overline{t})$ be a local minimum of $u - \phi$ in $U \times (s,S)$. Then, we have that $\left|\nabla\phi(\overline{x},\overline{t})\right| \geq f$, and since u is also a Barron-Jensen solution of (1.1),

$$\frac{\partial \phi}{\partial t}(\overline{x},\overline{t}) + H(\overline{x},\overline{t},\nabla\phi(\overline{x},\overline{t})) = 0.$$

By combining these estimates and (1.8) we further obtain that

$$-\frac{\partial \phi}{\partial t}(\overline{x},\overline{t}) - (a_1 + a_2 f) \geq 0,$$

and thus, u is a viscosity supersolution of

$$-\frac{\partial w}{\partial t} - (a_1 + a_2 f) = 0.$$

In an analogous way to the proof of Lemma 3.1 in [17] we now obtain that u is also a viscosity subsolution of (3.9) . □

This result can be used to estimate the decay of viscosity solutions:

Theorem 3.6. *Let $u \in C(\mathbb{R}^N \times (0,T))$ be a viscosity solution of* (1.1) *with $|\nabla u| \geq \kappa > 0$ in $U \times (s,S)$ in the viscosity sense for some open set U, and let the Hamiltonian H satisfy* (1.5), (1.8), (1.16) *and* (1.17). *Then the estimate* (3.8) *holds.*

Proof. Let the function $g : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} \kappa & \text{for } x \in U \\ 0 & \text{else} \end{cases} ,$$

and let $f_k : \mathbb{R}^N \to \mathbb{R}$ be a sequence of continuous functions such that f_k is identically zero in $\mathbb{R}^N - U$, $0 \leq f_k(x) \leq \kappa$ for $x \in U$, and $f_k \to g$ uniformly in any compact set not containing ∂U. Since u satisfies $|\nabla u| \geq f_k$ in the viscosity sense in $\mathbb{R}^N \times (s, S)$, we obtain from Lemma 3.5, that u is a viscosity subsolution of (3.9) with $f = f_k$. Moreover, the function $\tilde{u}_k$ defined by

$$\tilde{u}_k(x,t) := u(x,s) - (a_1 + a_2 f_k(x))(t-s).$$

is a viscosity supersolution of (3.9) with $\tilde{u}_k(.,s) = u(.,s)$. Hence, from Theorem 2.4 we obtain that $u \leq \tilde{u}_k$ in $\mathbb{R}^N \times (s,S)$, and the limit $k \to \infty$ yields (3.8). □

4 Properties of the Growth Front

Now we turn our attention to the growth front $\Gamma(t)$, the set $\Omega(t)$, and their geometrical characteristics. A first important question is the so-called *non-empty interior problem*, which is rather difficult to solve for general Hamilton-Jacobi equations. For special equations such as the mean-curvature equation, examples are known, where the evolving zero level sets of the viscosity solution create a nonempty interior in finite time (cf. e.g. [11, 5]). For the class of first-order equations we are considering, this equation can be answered positively due to a result of Barles et. al. [3] (cf. also [17] for a different proof):

Proposition 4.1. *Let the assumptions of Theorem 3.3 hold. If* $\mathcal{L}^N(\Gamma(0)) = 0$, *then*

$$\mathcal{L}^N(\Gamma(t)) = 0, \qquad \forall\, t \in [0,T]. \tag{4.1}$$

Moreover, the front $\Gamma(t)$ *is independent of the initial value, i.e., if* $u_{0,i}$, $i = 1,2$, *satisfy* $\{\ x \in \mathbb{R}^N \mid u_{0,i}(x) = 0\ \} = \Gamma(0)$ *for* $i = 1,2$, *then the sets* $\Gamma_1(t)$ *and* $\Gamma_2(t)$ *defined by* (1.2) *with* u *being the unique viscosity solution of* (1.1) *with initial value* $u_{0,1}$ *and* $u_{0,2}$, *respectively, are equal for all* $t \in [0,T]$.

As an immediate consequence of Corollary 3.2 we obtain a comparison of $\Omega(t)$ with the level sets of u_- and u_+, i.e,

$$\{\ x \in \mathbb{R}^N \mid u_-(x,t) \leq 0\ \} \subset \Omega(t) \subset \{\ x \in \mathbb{R}^N \mid u_+(x,t) \leq 0\ \} \tag{4.2}$$

holds for all $t \in (0,T)$, which well be deduced subsequently to derive lower and upper bounds for $\Omega(t)$. Moreover, from the nonnegativity of the Hailtonian, we may deduce the monotonicity of $\Omega(t)$ with respect to time, which shows that $\Gamma(t)$ really represents a growth front:

Proposition 4.2. *Under the assumptions of Theorem 2.4,the inclusion* $\Omega(s) \subset \Omega(t)$ *holds for* $s < t$. *Moreover, if either* $a_1 > 0$ *or* $a_1 = 0$ *and the assumptions of Theorem 3.3 (ii) are satisfied, then the growth front satisfies* $\Gamma(t) \cap \Gamma(s) = \emptyset$ *for* $s < t$.

Proof. Because of the nonnegativity of the Hamiltonian, the stationary function $\psi(.,t) := u(.,s)$ is a viscosity supersolution of (1.1) in the time interval $[s,t]$ with the same initial value as u at time t. Hence, by Theorem 2.4 we obtain that $u(.,s) = \psi(.,t) \geq u(.,t)$ and thus,

$$\Omega(s) = \{\ x \in \mathbb{R}^N \mid u(.,s) \leq 0\ \} \subset \{\ x \in \mathbb{R}^N \mid u(.,t) \leq 0\ \} = \Omega(t).$$

If $a_1 > 0$ we obtain that $\frac{\partial u}{\partial t} \leq -a_1 < 0$ in the viscosity sense in $\mathbb{R}^N \times (s,t)$ and consequently the set $\{ x \in \mathbb{R}^N \mid u(x,t) \leq -\alpha \}$ contains the sets $\Omega(s)$ for $\alpha < a_1(t-s)$, which implies

$$(\Gamma(s) \cap \Gamma(t)) \subset (\Omega(s) \cap \Gamma(t)) = \emptyset.$$

If $a_1 = 0$ and the assumptions of Theorem 3.3 (ii) are satisfied, then we obtain a lower bound estimate on $|\nabla u(x,t)|$ on $\{ x \in \mathbb{R}^N \mid -\delta \leq u(x,t) \leq \delta \}$ for δ sufficiently small and consequently obtain that $\frac{\partial u}{\partial t}$ is negative and bounded away from zero on this level set, which allows an analogous reasoning as in the first case. □

Another interesting property is the boundedness of the number of connected components of $\Omega(t)$ and $\Gamma(t)$. For a growth model without nucleation of new objects, it is obvious that the number of connected components cannot increase with respect to time (but it may decrease if objects merge). This statement is made rigorous by the following result:

Proposition 4.3. *Let the assumptions of Proposition 4.2 be satisfied and suppose that $\Omega(0)$ and $\Gamma(0)$ are compact sets with M connected components. Then, for all $t > 0$, the sets $\Omega(t)$ and $\Gamma(t)$ are compact sets with at most M connected components.*

Proof. The compactness of $\Omega(t)$ and $\Gamma(t)$ follows from the finite speed of propagation and the continuity of u. If $\Omega(s)$ or $\Gamma(s)$ has more than M connected components for some $s > 0$, then there exist a nonempty set U_1 and an open set U_2 with $U_1 \subset U_2$ such that $u(.,s) > 0$ in U_1, $u(.,0) < 0$ in U_2 and $u(.,t) \geq 0$ on ∂U_2 for all $t \in [0,s]$. Thus, the function $\psi \equiv 0$ is a viscosity subsolution of (1.18) in $U_2 \times [0,T]$ and $\psi(.,0) \geq u(.,0)$ as well as $\psi(x,t) \geq u(x,t)$ for $x \in \partial U$. Hence, by comparison we obtain that $0 = \psi \geq u$ in $U_2 \times [0,s]$, which contradicts $u(.,s) > 0$ in U_1. □

4.1 Regularity of the Growth Front

In the following part we are concerned with the local Lipschitz regularity of the growth front $\Gamma(t)$. To the author's knowledge, the only result for the level sets of u is due to Ley [17] who should that the set $\Gamma^\alpha(t)$ is locally Lipschitz for almost every α in an interval around $\alpha = 0$. Unfortunately, this statement gives no information on the growth front $\Gamma(t)$ itself, which need not be Lipschitz continuous even if the initial set $\Gamma(0)$ is arbitrarily smooth (cf. Figure 1 for an illustration). The counter-example given by Ley is the propagation of two distinct radial shapes via the eikonal equation, i.e., for the simple Hamiltonian $H(x,t,p) = |p|$. At some time $t_0 > 0$ the balls meet and the arising set $\Gamma(t_0)$ is not locally Lipschitz at the contact point. Nonetheless, the evolving curve $\Gamma(t)$ in this example is locally Lipschitz continuous $\mathcal{H}^{N-1}$-almost everywhere, for almost all $t \in [0,T]$, which is not a pecularity of this example as we shall prove below.

For this sake we introduce the arrival time

$$\tau(x) := \inf\{ t \in \mathbb{R}^+ \mid u(x,t) = 0 \} = \inf\{ t \in \mathbb{R}^+ \mid u(x,t) \leq 0 \}. \tag{4.3}$$

Due to the monotonicity in the evolution we obtain that the set $\{ t \in \mathbb{R}^+ \mid u(x,t) = 0 \}$ contains only a single element, so that τ is defined implicitely by the relation $u(x,\tau(x)) = 0$ and the level set $\Gamma(t)$ can be represented equivalently via

$$\Gamma(t) = \{ x \in \mathbb{R}^N \mid \tau(x) = t \}. \tag{4.4}$$

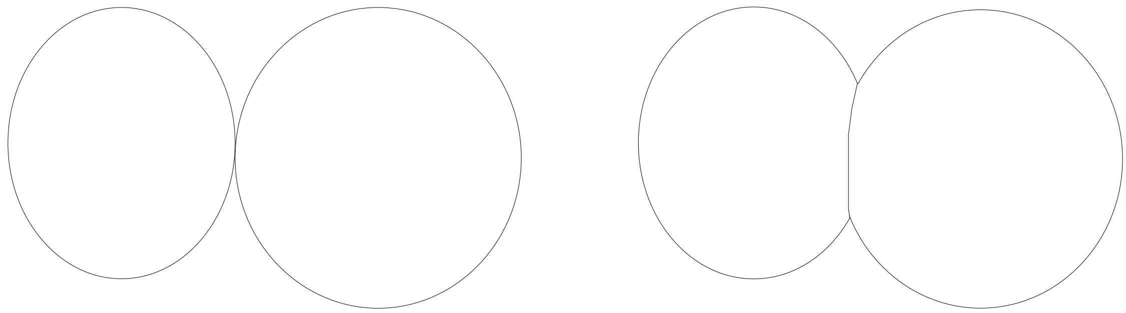

Figure 1: Schematic representation of a front with non-Lipschitz respectively local Lipschitz regularity.

This representation of the level set for monotonically advancing fronts has been used for modelling crystal growth by the minimal-time method (cf. [27]) and to develop the so-called *fast marching methods* for computing the evolution of growth fronts. Fast marching methods are based on solving the stationary Hamilton-Jacobi equation (cf. [1, 23, 25])

$$H(x, \tau, \nabla\tau) = 0, \qquad \text{in } \mathbb{R}^N - \Omega(0), \tag{4.5}$$

with the boundary condition $\tau = 0$ on $\Gamma(0)$ to obtain the arrival time as a viscosity solutions. The analysis of viscosity solutions for the stationary equation (and the equivalence of its viscosity solutions and the arrival time) is presently available only if the Hamiltonian is nonincreasing with respect to the second argument, which is a too strong assumption for the applications we have in mind. Therefore we will only use the definition as arrival time in the following and directly deduce its properties from the ones of the viscosity solution u of (1.1), (1.4).

In order to obtain a function τ defined on $\mathbb{R}^N - \Omega(t)$ we extend the Hamiltonian continuously to $\mathbb{R}^N \times (T, \infty)$ via $H(x, t, p) = H(x, T, p)$ for $t \geq T$ and consider the associated viscosity solution u defined on $\mathbb{R}^N \times \mathbb{R}^+$. We start with results on the well-definedness and local Lipschitz continuity of the function τ:

Lemma 4.4. *Let the assumptions of Proposition 4.2 hold, and let $\Omega(0)$ be a compact set with $\Gamma(0) = \partial\Omega(0)$ such that the signed distance function σ_{Γ_0} is Lipschitz continuous. Then the arrival time defined by (4.3) satisfies $0 < \tau(x) < \infty$ for all $x \in \mathbb{R}^N - \Omega(0)$.*

Proof. Let τ^+ and τ^- be the arrival times associated with the functions u_-^0 and u_+^0. Due to Corollary 3.2, we have $\tau^+ \leq \tau \leq \tau^-$ on $\mathbb{R}^N - \Omega(0)$. Since the growth front is independent of the initial value in the sense of Proposition 4.1, we may assume without restriction of generality that u_-^0 and u_+^0 are the solutions with initial value $\sigma_{\Gamma(0)}$, which satisfies $\left|\nabla\sigma_{\Gamma(0)}\right| = 1$ in the viscosity sense in $\mathbb{R}^N$. As we have seen at the end of Section 3, this implies that $\left|\nabla u_\pm^0\right| = 1$ in the viscosity sense, and hence, we obtain that

$$\frac{\partial u_-^0}{\partial t} = -(a_1 + a_2), \qquad \frac{\partial u_+^0}{\partial t} = -(A_1 + A_2)$$

in the viscosity sense. Thus, the associated arrival times satisfy

$$\tau^-(x) = \frac{\sigma_{\Gamma(0)}(x)}{a_1 + a_2}, \qquad \tau^+(x) = \frac{\sigma_{\Gamma(0)}(x)}{A_1 + A_2},$$

for $x \in \mathbb{R}^N - \Omega(0)$, which gives a lower and an upper bound for $\tau(x)$. □

Proposition 4.5. *Let the assumptions of Theorem 3.3 (ii) and Lemma 4.4 be satisfied, then τ is locally Lipschitz continuous in $\Omega(T) - \Omega(0)$, and its norm in $W^{1,\infty}(\Omega(T) - \Omega(0))$ depends on T, H, and u_0 only.*

Proof. Let $x \in \Omega(T)1 - \Omega(0)$.. Due to Theorem 3.3, there exists a neighborhood U of x, a time $t_0 > \tau(x)$, and a constant $c_0 = e^{-\frac{\gamma T}{2}}\sqrt{\frac{C}{2}}$ such that $|\nabla u| \geq c_0$ in $U \times (\tau(x), t_0)$. Hence,

$$\frac{\partial u}{\partial t} \leq -a_1 - a_2 c_0 < 0, \qquad \text{in the viscosity sense in } U \times (s, t_0).$$

Now consider the set $P(x) = U \cap \{\ y \in \Omega(T) \mid t_0 \geq \tau(y) \geq \tau(x)\ \}$. If U is sufficiently small, then $u(y, \tau(x)) \leq (a_1 + a_2 c_0)(t_0 - \tau(x))$ for $y \in P(x)$ and hence, with the estimate for $\frac{\partial u}{\partial t}$ we may deduce that $u(y, \tau(x) + \delta) \leq 0$ for $\delta = \frac{u(y,\tau(x))}{a_1 + a_2 c_0}$. Thus, noticing that $u(x, \tau(x)) = 0$, we obtain the estimate

$$\tau(y) - \tau(x) \leq \frac{1}{a_1 + a_2 c_0}\left(u(y, \tau(x)) - u(x, \tau(x))\right)$$

and from Theorem 2.3 we may deduce the existence of a constant $c_1 = c_1(T, H, u_0)$ such that $\tau(y) - \tau(x) \leq c_1 |x - y|$. Since the constant c_1 does not depend on x we may conclude that $|\nabla \tau| \leq c_1$ almost everywhere in $\Omega(T) - \Omega(0)$. □

In order to analyze the regularity of the front $\Gamma(t)$, we introduce the concept the subdifferential for locally Lipschitz functions $f : \mathbb{R}^N \to \mathbb{R}$, defined by (cf. [9] for a detailed discussion)

$$\partial_C f(x) = \{\ h \in \mathbb{R}^N \mid \limsup_{y \to x, \epsilon \downarrow 0} \frac{f(y + \epsilon p) - f(y)}{\epsilon} \geq \langle h, p \rangle, \forall\ p \in \mathbb{R}^N\ \}. \tag{4.6}$$

Due to Rademacher's Theorem, a locally Lipschitz continuous function is differentiable for almost every x and the identity $\partial_C f(x) = \{\nabla f(x)\}$ holds at such points. By generalization of the implicit function theorem to locally Lipschitz functions, we will show in the following that the growth front is locally Lipschitz continuous at a point x, if $0 \notin \partial_C \tau$. For this sake we introduce the following notations:

Definition 4.6. We set

$$\mathcal{S}_0 := \{\ x \in \cup_{t \in [0,T]} \Gamma(t) \mid 0 \in \partial_C \tau(x)\ \} \tag{4.7}$$

and

$$\mathcal{S} := \{\ x \in \cup_{t \in [0,T]} \Gamma(t) \mid \Gamma(t) \text{ is not Lipschitz at } x\ \} \tag{4.8}$$

An important relation between $\mathcal{S}$ and $\mathcal{S}_0$ is the following result:

Lemma 4.7. *Let $\tau \in C(\Omega(T) - \Omega(0))$ be locally Lipschitz continuous. Then $\mathcal{S} \subset \mathcal{S}_0$.*

Proof. The proof can be carried out in an analogous way to the proof of Theorem 5.4 in [17]. □

Now we are in position to prove the main result of this section:

Theorem 4.8. *Under the assumptions of Proposition 4.5,*

$$\mathcal{H}^{N-1}(\mathcal{S}\cap\Gamma(t)) = 0, \qquad \text{for almost every } t \in (0,T).$$

Proof. Let Σ denote the set of points $x \in \Omega(T)-\Omega(0)$ such that τ is differentiable at x. From Rademacher's Theorem, we may conclude that $\mathcal{L}^N(\mathcal{S}-\Sigma)=0$ and $\partial_C\tau(x)=\{\nabla\tau(x)\}=\{0\}$ for $x\in\mathcal{S}\cap\Sigma$. Hence, with the coarea formula and the we obtain

$$\begin{aligned}\int_0^T \mathcal{H}^{N-1}(\mathcal{S}\cap\Gamma(t))\ dt &= \int_{\mathcal{S}} |\nabla\tau|\ d\mathcal{L}^N = \int_{\mathcal{S}-\Sigma} |\nabla\tau|\ d\mathcal{L}^N \\ &\le \mathcal{L}^N(\mathcal{S}-\Sigma)\,\|\tau\|_{W^{1,\infty}} = 0.\end{aligned}$$

which implies the assertion. □

Finally, the boundedness of τ in $W^{1,\infty}$ allows to deduce an upper bound on the Hausdorff-measure of the propagating front, when interpreted as an L^1-function of the time variable:

Theorem 4.9. *Under the assumptions of Proposition 4.5, the estimate*

$$\int_0^T \mathcal{H}^{N-1}(\Gamma(t))\ dt \le c\ (\mathit{diam}\ \Omega(0)+(A_1+A_2)T)^N \tag{4.9}$$

holds for some constant $c\in\mathbb{R}^+$.

Proof. From the coarea formula and the representation (4.4) of $\Gamma(t)$ we obtain that

$$\int_0^T \mathcal{H}^{N-1}(\Gamma(t))\ dt = \int_{\cup_t\Gamma(t)} |\nabla\tau(x)|\ dx \le c_0\ \mathcal{L}^N(\Omega(T)),$$

where $c_0=\|\tau\|_{W^{1,\infty}}$. From the proof of Lemma 4.4 we observe that $\tau(x)>T$ for $\sigma_{\Gamma_0}(x) > (A_1+A_2)T$ and thus,

$$\text{diam}\ \Omega(T) \le \text{diam}\ \Omega(0) + 2(A_1+A_2)T,$$

which implies (4.9). □

References

[1] D.Adalsteinsson, R.Kimmel, R.Malladi, J.A.Sethian, *Fast marching methods for computing the solutions to static Hamilton–Jacobi equations*, CPAM Report 667 (University of California, Berkeley, 1996).

[2] G.Barles, *Solutions de Viscosité des Équations de Hamilton-Jacobi* (Springer, Berlin, 1994).

[3] G.Barles, H.M.Soner, P.E.Souganidis, *Front propagation and phase-field theory*, SIAM J. Cont. Optim. **31** (1993), 439-469.

[4] E.N.Barron, R.Jensen, *Semicontinuous viscosity solutions of Hamilton-Jacobi equations with convex Hamiltonians*, Commun. PDE **15** (1990), 1713-1742.

[5] G.Bellettini, M.Novaga, M.Paolini, *An example of three dimensional fattening for linked space curves evolving by curvature*, Comm. Partial Differential Equations, **23** (1998), 1475-1492.

[6] M.Burger, *Growth of multiple crystals in polymer melts,* Preprint (2001), and submitted.

[7] M. Burger, V. Capasso, *Mathematical modelling and simulation of non-isothermal crystallization of polymers*, Math. Models and Meth. in Appl. Sciences **11** (2001), 1029-1054.

[8] Y.G.Chen, Y.Giga, S.Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differ. Geom. **33** (1991), 749-786.

[9] F.H.Clarke, *Optimization and Nonsmooth Analysis* (SIAM, Philadelphia, 1990).

[10] M.G.Crandall, P.L.Lions, *On existence and uniqueness of solutions of Hamilton-Jacobi equations*, Nonlinear Analysis **10** (1986), 353-370.

[11] L.C.Evans, J.Spruck, *Motion of level sets by mean curvature. I*, J. Differ. Geom. **33** (1991), 635-681.

[12] L.C.Evans, J.Spruck, *Motion of level sets by mean curvature. II*, Trans. Am. Math. Soc. **330** (1992), 321-332.

[13] L.C.Evans, J.Spruck, *Motion of level sets by mean curvature. III*, J. Geom. Anal. **2** (1992), 121-150.

[14] L.C.Evans, J.Spruck, *Motion of level sets by mean curvature. IV*, J. Geom. Anal. **5** (1995), 79-116.

[15] H.Federer, *Geometric Measure Theory* (Springer, Berlin, Heidelberg, New York, 1969).

[16] A.Friedman, J.L.Velazquez, *A free boundary problem associated with crystallization of polymers in a temperature field*, Indiana Univ. Math. Journal **50**, 4 (2001).

[17] O.Ley, *Lower bound gradient estimates for first-order Hamilton-Jacobi equations and applications to the regularity of propagating fronts*, Adv. Differ. Equ. **6** (2001), 547-576.

[18] J.L.Lions, E.Magenes, *Non-Homogeneous Boundary Value Problems and Applications* (Springer, Berlin, Heidelberg, New York, 1972)

[19] P.L.Lions, *Generalized Solutions of Hamilton-Jacobi Equations* (Pitman, Boston, London, Melbourne, 1982).

[20] J.M.Morel, S.Solimini, *Variational Methods in Image Segmentation with 7 Image Processing Experiments* (Birkhäuser, Basel, 1994).

[21] S.Osher, R.P.Fedkiw, *Level set methods: An overview and some recent results*, J. Comp. Phys. **169** (2001), 463-502.

[22] S.Osher, J.A.Sethian, *Fronts propagating with curvature-dependent speed: algorithms based on Hamilton–Jacobi formulations*, J. Comp. Phys. **79** (1988), 12–49.

[23] J.A.Sethian, *Level Set Methods. Evolving Interfaces in Geometry, Fluid Mechanics, Computer Vision, and Materials Science* (Cambridge Univ. Press, Cambridge, 1996).

[24] J.A.Sethian, *Fast marching level set methods for three-dimensional photolithography development*, Proceedings of the SPIE 1996 International Symposium on Microlithography (Santa Clara, California, 1996).

[25] J.A.Sethian, *Fast marching methods*, SIAM Review **41** (1999), 199-235.

[26] R.E.Showalter, *Monotone Operators and Nonlinear Partial Differential Equations in Banach Space* (AMS Mathematical Surveys and Monographs, Providence, Rhode Island, 1997).

[27] J.E.Taylor, J.W.Cahn, C.A.Handwerker, *Geometric models of crystal growth*, Acta metall. mater. **40** (1992), 1443-1472.